HISTORY DETERMINISM

Lecture 4: From 2-tokens to k-tokens, and then to joker.

Reference: Sections 3.2.1 and 3.2.2 of Keya Prakash's PhD thesis

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Plan for the lecture:

Part 1: Eve wins 2-token game $\Rightarrow$ Eve wins k-token game $\forall k \geq 1$

Part 2: Definition of joker games and examples

Part 3: Eve wins 2-token game $\Rightarrow$ Eve wins joker game

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Part 1: 2-tokens vs k-tokens.

We start with a simpler direction, going from $k \geq 3$ to 2.

Lemma 1: Let $\mathcal{A}$ be a parity automaton and let $k \geq 3$.
Eve wins k-token game on $\mathcal{A}$ $\Rightarrow$ Eve wins 2-token game on $\mathcal{A}$.

Proof: Assume Eve wins k-token game on $\mathcal{A}$, for some $k \geq 3$.

- Let σ_k be a winning strategy.
- Now: consider the 2-token game.
- Eve adds $(k-2)$ extra tokens in which she simply copies her token.
- Eve can then use σ_k in this 2-token game with the 'dummy' $k-2$ tokens.
- This will be a winning strategy in the 2-token game. $\square$

Let us now look at the more non-trivial direction — from 2 to k.

Firstly if $k = 1$, then by a similar argument as in Lemma 1, we can show that Eve wins the 1-token game.

The next lemma considers the case when $k \geq 3$. This was proved by Bagnol and Kuperberg in FSTTCS '2018.

Lemma 2: Let $\mathcal{A}$ be a parity automaton.

Eve wins 2-token game $\Rightarrow$ Eve wins k-token game $\forall k \geq 3$

Proof: Assume Eve wins 2-token game. We will show that Eve wins the 3-token game. The argument for greater k is analogous.

- Let σ_2 be a winning strategy for Eve in the 2-token game.
- In the 3 token game, Adam has 3-tokens, say 1, 2, 3.
- Eve maintains a memory token where she plays strategy σ_2 w.r.t the second and third token of Adam.
- In her token, she plays σ_2 w.r.t. Adam's first token and her memory token.

Formally: suppose current configuration is:

Eve's token: p

Memory token: p'

Adam's token 1: q_1

Adam's token 2: q_2

Adam's token 3: q_3

+ Memory of σ_2 w.r.t. Adam 2 & Adam 3 : m_{23}

Memory of σ_2 w.r.t. Adam 1 + memory token : m_{10}

- When Adam plays letter 'a' in this configuration.
  - 1. Eve moves memory token to:

$$p'' = \sigma_2 (p' ; q_2 , q_3 , m_{23})$$

Eve updates memory m_{23}.

  - 2. Then, Eve moves her token to:

$$\sigma_2 (p ; q_1 , p'' , m_{10})$$

Eve updates memory m_{10}

This explains the finite-memory strategy of Eve in the 3-token game.

It is easy to see that this strategy is winning for Eve in the 3-token game.

Part 2: Joker games

Definition: Let $\mathcal{A}$ be a parity automaton. The joker game starts with an Eve's token and Adam's token both initially at q_0.

- The game proceeds in infinitely many rounds, and in round i, where Eve's token is at q_i and Adam's token is at p_i:

- 1. Adam selects a letter $a_i \in \Sigma$.
- 2. Eve selects a transition $q_i \xrightarrow{a_i : c_i} q_{i+1}$
- 3. Adam either moves his token along a transition

$$p_i \xrightarrow{a_i : c_i} p_{i+1}$$

or plays Joker and selects a transition:

$$q_i \xrightarrow{a_i : c_i} p_{i+1}$$

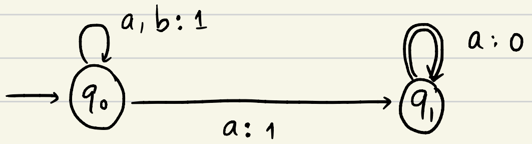
and places his token at p_{i+1}.

The new position is (q_{i+1}, p_{i+1}) where round $i+1$ begins.

Eve wins a play: if Adam's plays finitely many Jokers and if Adam's sequence of transitions satisfies the parity condition, then Eve's token also satisfies the parity condition.

Example: Consider

$\mathcal{A}_1$:



Adam wins Joker game on $\mathcal{A}_1$ as follows:

- Adam plays 'a' in Round 0 and irrespective of what Eve does, moves to q_1.
- Adam continues to play 'a', and keeps the token at q_1 until Eve remains in q_0.

If Eve remains in q_0, then Adam wins.

- Suppose at Round j, we have:

  Eve: $q_0 \longrightarrow q_1$ (Eve moving to q_1).

  Then in that round, Adam plays Joker and does the following:

  Adam: $q_0 \longrightarrow q_0$.

- Now, Adam plays a 'b' followed by a^ω where he goes to q_1.
- Eve cannot extend token on a 'b' and hence loses.

Proposition: Joker games are finitely-determined.

Additionally, if Eve wins a Joker game, then she has a positional winning strategy.

Proof Sketch:

Joker games can be modeled as 2-dimensional parity games for which we have finite-memory determinacy. Moreover, for 2-dimensional parity games, Eve has a uniform positional winning strategy in her winning region.

See Chapter 2, Theorem 2.2 in Keya Prakash's Ph.D thesis for more details.

Part 3: Eve wins 2-token game implies Eve wins Joker game.

As a preliminary step, we show that if Adam wins a Joker game, then there is a uniform bound on the number of Joker moves he needs to play to win the game.

Lemma 3: Let $\mathcal{A}$ be a parity automaton on which Adam wins the Joker game. Let σ_J be a finite-memory winning strategy for Adam.

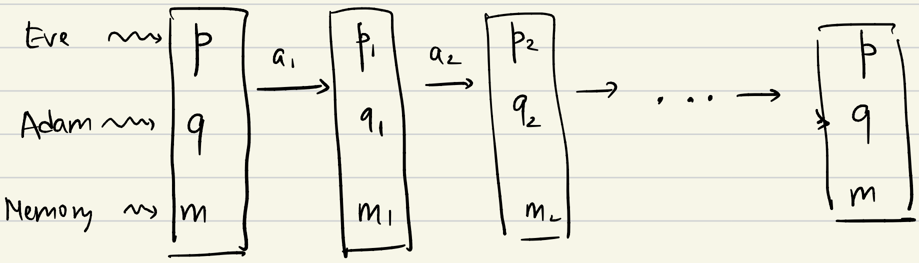
Let $K = 1 +$

(Size of arena of Joker game) $\times$ (size of Adam's memory)

Then every play following σ_J contains at most K Joker moves.

Proof:

Consider a play where the same game position and memory repeats.



- Suppose there is a Joker move in ρ. Then at (p, q, m) Adam will choose the same letter a_1. Eve can then move her token to p_1.
- Then Adam goes to q_1, m_1 as per σ_J.
- Essentially, Eve can force the same ρ to repeat indefinitely.
- Adam loses such a play since there would be infinitely many Joker moves.

Now, If a play following σ_J has $\geq K+1$ Joker moves, it will have a cycle of the above form, with a Joker move in between.

This contradicts σ_J being a winning strategy for Adam.

Lemma 4: Let $\mathcal{A}$ be a parity automaton.

If Eve wins the 2-token game on $\mathcal{A}$, then she wins the Joker game on $\mathcal{A}$.

Proof:

We prove the contrapositive.

- If Adam wins the Joker game, then Adam wins the 2-token game.

Here we have used the fact that both Joker games & token games are determined.

- Assume Adam wins the Joker game. By Lemma 3, Adam's winning strategy makes at most k Joker moves for some k determined by the game's syntax.

- Let σ_J be Adam's winning strategy in the Joker game.

- Consider the $k+1$ token game.

- Adam plays σ_J on the first token and copies Eve's run in all the other tokens.

- At the first Joker move, Adam follows σ_J at the second token.

  Copies Eve's run in tokens 3 to k and makes dummy move at token 1.

- At the second Joker move, Adam follows σ_J at the third token

  and so on...

- Eventually there will be some token of Adam which will give an actual run of the automaton.

→ This run will satisfy the parity condition, whereas Eve's run will not, as σ_J is a winning strategy of the Joker game.

→ This proves that Adam wins the $k+1$ token game, and hence the 2-token game (due to Lemma 2).