Aug 7, 2026

HISTORY DETERMINISM

Lecture 3: Characterizing HD-ness in Büchi and coBüchi automata using token games & joker games - Part I

Reference: Ph.D. Thesis of Keya Prakash

To get more insights into the notion of history-determinism, the research community has introduced two new games:

- k-token games [Bagnol, Kuperberg '2018]
- Joker games [Kuperberg & Skrzypczak '2015]

Using these games, we now have the following characterizations of HD-ness on different types of automata.

Characterizing HD on Büchi & coBüchi automata

Theorem: Let $\mathcal{A}$ be an ω-automaton with either a coBüchi or a Büchi acceptance condition.

The following are equivalent.

- 1. $\mathcal{A}$ is HD.
- 2. Eve wins the 2-token game.
- 3. Eve wins the joker game

For coBüchi this was proved in [Boker, Kuperberg, Lehtinen, Skrzypczak '2020]

For Büchi this was proved in [Acharya, Jurdzinski, Prakash '2024]

We follow the presentation in Keya Prakash's Ph.D. thesis.

Our goal is to understand this theorem across several lectures.

Plan for today's lecture:

Part 1: Revisit the letter game and show that it is finitely determined

Part 2: Introduce the k-token game

Part 3: Some examples of 1-token and 2-token games

Suggested reading: For background on ω-automata and games please see chapter 2 of Keya's thesis

- chapter 11 of Bojańczyk's "Automata Toolbox" talks about infinite duration games and the Büchi-Landweber theorem

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Notation: In general, an ω-automaton is $(Q, \Sigma, I, \Delta, \alpha)$ where α is an "accepting condition". Here are some specific conditions which will be useful to us.

Büchi automaton: $(\Sigma, Q, I, \Delta, F)$ $F \subseteq \Delta$ a subset of transitions

Run ρ is accepting if: $\text{Inf}(\rho) \cap F \neq \phi$

set of transitions occurring infinitely often in ρ

coBüchi automaton: $(\Sigma, Q, I, \Delta, F)$ $F \subseteq \Delta$

Run ρ is accepting if: $\text{Inf}(\rho) \subseteq F$

Muller automaton: $(\Sigma, Q, I, \Delta, \mathcal{F})$ $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$

each $F_i \subseteq \Delta$

Run ρ is accepting if $\text{Inf}(\rho) = F_i$ for some i

Part 1: Revisiting the letter game HD($\mathcal{A}$)

Let $\mathcal{A}$ be an ω-automaton. The letter game (also called HD-game) on $\mathcal{A}$, denoted as HD($\mathcal{A}$) is played between Adam & Eve.

- It is infinite duration
- At any finite number of moves: Adam has chosen a word in Σ^+ and Eve has chosen a run of this word on $\mathcal{A}$.

- At round $i \geq 0$
  - Adam chooses $a_i \in \Sigma$
  - Eve chooses a transition $\tau_i \in \Delta$ over this letter a_i and which extends her run so far.

    ↳ at round 0, Eve chooses a transition from an initial state

Winning objective: An infinite play is of this form:

$$\sigma = \begin{pmatrix} a_0 \\ \tau_0 \end{pmatrix} \begin{pmatrix} a_1 \\ \tau_1 \end{pmatrix} \begin{pmatrix} a_2 \\ \tau_2 \end{pmatrix} \cdots$$

Let $w = a_0 a_1 \ldots$ and $\tau = \tau_0 \tau_1 \ldots$

$$\boxed{\text{Eve wins } \sigma \text{ if either } w \notin \mathcal{L}(\mathcal{A}) \text{ or } \tau \text{ is an accepting run}}$$

Notice that by design Eve's run would be a run of $\mathcal{A}$ over w.

If Eve is unable to extend her run at some round, she loses.

Let $L^{HD}_{Eve} = \{ \sigma \text{ as above} \mid \sigma \text{ is winning} \}$

Finite-memory determinacy of letter games:

We claim that L^{HD}_{Eve} is an ω-regular language.

To prove this, we will build a Muller automaton for L^{HD}_{Eve}

- Notice that the infinite plays are of the following form:

$$\begin{pmatrix} a_0 \\ \tau_0 \end{pmatrix} \begin{pmatrix} a_1 \\ \tau_1 \end{pmatrix} \cdots$$

where $a_i \in \Sigma$, $\tau_i \in \Delta$ (of the given automaton $\mathcal{A}$).

So the alphabet is $\Sigma \times \Delta$

- Let $\mathcal{B}$ be a deterministic Muller automaton s.t. $\mathcal{L}(\mathcal{B}) = \mathcal{L}(\mathcal{A})$.

$$\mathcal{B} = (Q^B, \Sigma, I^B, \Delta^B, \mathcal{F}^B)$$

Assume $\mathcal{A}$ to be a Muller automaton (as this is more general than Büchi & coBüchi)

$$\mathcal{A} = (Q^A, \Sigma, I^A, \Delta^A, \mathcal{F}^A)$$

- Construct automaton C^{HD}_{Eve} as follows:
  - States $Q^C = Q^B \times Q^A$
  - Alphabet $\Sigma^C = \Sigma \times \Delta$
  - Initial states $I^C = I^B \times I^A$
  - Transitions Δ^C:

$$(p^B, p^A) \xrightarrow{\binom{a}{\tau}} (q^B, q^A) \quad \text{if}$$

$$p^B \xrightarrow{a} q^B \quad \text{in } \mathcal{B}$$

$$\text{and} \quad \tau = (p^A, a, q^A)$$

  - Acceptance condition $\mathcal{F}^C$: defined in next page

* For a transition $\theta : (p^B, p^A) \xrightarrow{\binom{a}{\tau}} (q^B, q^A)$, we define

$$\pi^B(\theta) = (p^B, a, q^B)$$

$$\pi^A(\theta) = \tau = (p^A, a, q^A)$$

* For a subset $S \subseteq \Delta^C$ of transitions of C, define

$$\pi^B(S) = \{ \pi^B(\theta) \mid \theta \in S \}$$

$$\pi^A(S) = \{ \pi^A(\theta) \mid \theta \in S \}$$

The acceptance condition $\mathcal{F}^C$ is given as follows:

$$\mathcal{F}^C = \{ S \subseteq \Delta^C \mid \pi^B(S) \in \mathcal{F}^B \text{ implies } \pi^A(S) \in \mathcal{F}^A \}$$

Part 2: The k-token game

- Given a non-deterministic ω-automaton $\mathcal{A}$.
- The k-token game is played between Adam & Eve.
- Eve has 1 token and Adam has k tokens.
- Initially, all the tokens are on some initial state of $\mathcal{A}$.

  Eve's token is at q_0.

  Adam's j^{th} token is at p_0^j

- The game proceeds in infinitely many rounds. Before round $i \geq 0$,

  Eve's token is at q_i and Adam's j^{th} token is at p_i^j

- At round $i \geq 0$.
  - 1. Adam chooses a letter $a_i \in \Sigma$
  - 2. Eve selects a move $q_i \xrightarrow{a_i} q_{i+1}$
  - 3. Adam selects a move for each of his tokens

$$p_i^j \xrightarrow{a_i} p_{i+1}^j$$

Winning plays: In a play, Adam constructs a word and k runs of $\mathcal{A}$ on this word,

and Eve generates a run of $\mathcal{A}$ on the word.

Eve wins a play if the following condition is satisfied:

- if at least one of the runs of Adam's tokens is accepting,

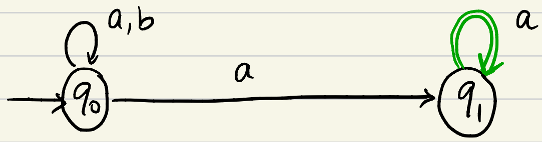
  then Eve's run is accepting.

Otherwise Adam wins the play.

Exercise: Show that k-token games are finitely determined.

Part 3: Examples of token games

Let $\mathcal{A}_1$ be:



coBüchi acceptance with the green transition being the accepting transition.

Eve wins the 1-token game on $\mathcal{A}_1$

Eve's strategy is as follows:

→ At the start of round i, if Adam's token is at q_0, play $q_0 \xrightarrow{*} q_0$ at round i.

→ If Adam moves to q_1 at round i, Eve moves to q_1 at round $i+1$.

This is a winning strategy for Eve.

- If Adam remains in q_0, then Adam's run is not accepting and hence Eve wins.

- Once Adam moves to q_1, then he can only play 'a' and remain in q_1. This will be an acc. run.

  However, by the strategy, Eve will move to q_1 as well and hence Eve wins the 1-token game on $\mathcal{A}_1$

Adam wins the 2-token game on $\mathcal{A}_1$:

Here is Adam's strategy:

- Adam keeps playing 'a', keeps token 1 at q_0 and moves token 2 to q_1 at the first round.

- Suppose Eve moves her token to q_1 at round i.

- At $i+1$, Adam plays a 'b' and then continues to play 'a'.

This is a winning strategy for Adam:

- If Eve remains in q_0, she loses.
- Once Eve moves to q_1, she cannot extend her token anymore, whereas Adam manages to produce an accepting run.