HISTORY DETERMINISM

Lecture 2: History determinism coincides with good-for-gameness for ω-regular automata

References: 1. When a little non-determinism goes a long way: An introduction to history-determinism
Udi Boker & Karoliina Lehtinen
ACM, Siglog News, Volume 10, Issue 1 (2023)

2. Chapter 11 of the book "An Automata Toolbox" (December '25 version)
by Mikołaj Bojańczyk

Plan for the lecture:

Part 1: Formal definition of history-determinism, the "letter game"

Part 2: Good-for-games automata

2.1 Infinite duration games with ω-regular winning objective

2.2. Büchi-Landweber theorem

2.3. GFG automata - definition

Part 3: HD $\equiv$ GFG for ω-regular automata

Part 1: History determinism - formal definition

Σ: a finite alphabet

Automaton $\mathcal{A}$: $(\Sigma, Q, I, \delta, \alpha)$

Q : finite set of states

$I \subseteq Q$: initial states

$\delta : Q \times \Sigma \to 2^Q$ transition relation

α : acceptance condition

- varies depending on condition
- α could be a subset of states or a subset of transitions

- $\mathcal{A}$ is deterministic if $|\delta(q,a)| \leq 1$ $\forall q \in Q$ $\forall a \in \Sigma$

Letter game: HD($\mathcal{A}$)

- Letter game on an automaton $\mathcal{A}$
- Two-player win-lose game between Adam and Eve

- Adam builds an infinite word 'w' step by step and Eve builds a run of $\mathcal{A}$ on w.

- At turn i:
  - Adam chooses a letter a_i from Σ
  - Eve chooses a transition τ_i of $\mathcal{A}$ over a_i

- In the limit, a play consists of:

  a word $w = a_0 a_1 a_2 \ldots$

  and a sequence of transitions $\tau = \tau_0 \tau_1 \tau_2 \ldots$

Eve wins the play if

- either $w \notin L(\mathcal{A})$
- or τ is an accepting run of $\mathcal{A}$ over w

$\mathcal{A}$ is history - deterministic if Eve has a winning strategy in the letter game over $\mathcal{A}$, denoted as $HD(\mathcal{A})$

Strategy of Eve can be seen as:

$$r : \Sigma^* \times \Sigma \longrightarrow \Delta$$

where Δ is the set of transitions of $\mathcal{A}$

The strategy that witnesses HD is called resolver

Part 2: Good-for-Games automata

- Our next goal is to show that history-determinism coincides with a notion called
  Good-for-Gameness introduced by Henzinger & Piterman '06
- To understand this, we need an interlude on infinite duration games and the Büchi-Landweber theorem
  ↳ We recall these notions from Bojanczyk's book.

Part 2.1: Infinite duration games with ω-regular winning objectives

Games: Two players 0 & 1

- a directed graph, not necessarily finite, whose vertices are called positions
- a distinguished initial position
- partition of positions into those controlled by Player 0 and those controlled by Pl. 1
- a labeling of edges by a finite alphabet Σ.
- Winning condition: a function from $\Sigma^\omega \to \{0,1\}$

Play:
- Begins at initial position
- Player who controls current position chooses an outgoing edge
- Play moves to the new position corresponding to the target of the edge.
- If current position is a dead-end, player controlling that position loses.
- Otherwise play continues for ever: $v_0 \xrightarrow{a_0} v_1 \xrightarrow{a_1} v_2 \xrightarrow{a_2} \cdots$

- The sequence of edge labels $a_0 a_1 a_2 \ldots$ is a word in Σ^ω

Play is winning for player i if winning condition associates $a_0 a_1 a_2 \ldots$ to i.

Strategy for Player i:

- a function which takes as input a history of the play seen so far that ends in a position controlled by i and outputs an outgoing edge from this position
- Strategy is winning for i if every play conforming to this strategy is winning for Player i

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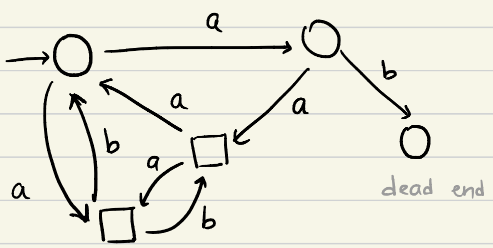
Example:

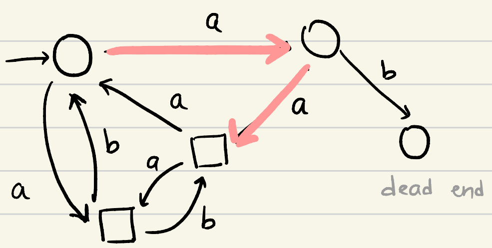


○ : Pl. 0

□ : Pl. 1

Winning condition: Pl. 0 wins if 'a' appears infinitely often

Winning strategy for Pl. 0.



Determinacy: A game is determined if one of the players has a winning strategy

Finite - memory strategies:

- Consider a game with positions V
- A finite memory strategy for Player i with memory M is given by:
  - a deterministic automaton with states M and alphabet V
  - for every position $v \in V$ controlled by i, a function f_v from M to neighbours of v

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- Suppose current position is v & current memory is m

$$(v, m)$$

- Apply $f_v(m)$. Let $v' = f_v(m)$

$$(v, m)$$
$$\downarrow$$
$$(v' = f_v(m), \quad)$$

- Then apply the transition $m \xrightarrow{v'} m'$

$$(v, m)$$
$$\downarrow$$
$$(v', m')$$

$v' = f_v(m)$

$m \xrightarrow{v'} m'$

Part 2.2: Büchi - Landweber Theorem

Let Σ be finite and let

$$Win: \Sigma^{\omega} \longrightarrow \{0,1\} \text{ be } \omega\text{-regular}$$

ie., inverse image of 0 (and 1) is recognized by a deterministic Muller automaton.

For every game with winning condition Win, one of the players has a finite - memory winning strategy

Proof-sketch of Büchi-Landweber theorem:

Two ingredients:

Lemma 1: For every deterministic Muller automaton, there is an equivalent deterministic parity automaton.

Lemma 2: For every parity game, one of the players has a memoryless winning strategy.

Consider a game G whose winning condition for player 0 is given by a deterministic parity automaton $\mathcal{A}$

$$\text{Game } G \; + \; \text{det. parity automaton } \mathcal{A}$$
$$\downarrow$$
$$\text{Product game } G \times \mathcal{A}$$

$$(v, q) \xrightarrow{a} (v', q') \quad \text{if } v \xrightarrow{a} v' \text{ in } G \text{ and } q \xrightarrow{a} q' \text{ in } \mathcal{A}$$

- (v, q) belongs to Pl. i if v belongs to Pl. i
- Winning condition in $G \times \mathcal{A}$: parity

The following are equivalent: (Exercise)

- 1. player i wins from position v in ω-regular game G
- 2. player i wins from (v, q_0) in the parity game $G \times \mathcal{A}$ where q_0 is the initial state q $\mathcal{A}$.

Note: The $1 \Rightarrow 2$ direction uses the fact that $\mathcal{A}$ is deterministic.

Part 2.3: Good - for - Games automata

An ω-regular automaton $\mathcal{A}$ is GFG if

for every game G with winning condition for Pl. 0 given by $L(\mathcal{A})$

Player i wins from position v in G

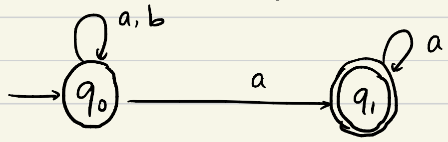
$\Updownarrow$

Player i wins from position (v, q_0) in $G \times \mathcal{A}$
where q_0 is initial state of $\mathcal{A}$

Winning condition of $G \times \mathcal{A}$ is same as acceptance condition of $\mathcal{A}$.

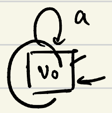
Exercise: Is the following automaton GFG ?

$\mathcal{A}$:



acc. condition:
eventually run remains in q_1

G:

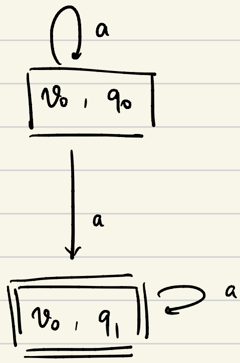


v_0 belongs to Pl. 1

winning condition $L(\mathcal{A})$

Clearly Pl. 0 wins

$G \times \mathcal{A}$:



Winning condition for Pl. 0 is to eventually reach and remain in $\boxed{v_0, q_1}$

Clearly Pl. 1 has a strategy to remain in (v_0, q_0) and hence wins $G \times \mathcal{A}$.

$\therefore$ $\mathcal{A}$ is not GFG.

Part 3: HD coincides with GFG for ω-regular automata.

Lemma 3: Let $\mathcal{A}$ be an HD automaton with an ω-regular winning condition.

Then $\mathcal{A}$ is GFG.

Lemma 4: Let $\mathcal{A}$ be an ω-regular automaton which is GFG.

Then $\mathcal{A}$ is HD.

The proof of these two lemmas is left as an assignment and will be discussed in class later.