

VASS REACHABILITY

Lecture 3 : Ackermann hardness

Reference: Reachability in VAS is Ackermann-complete

↳ Czerwiński & Drlíkowski (FOCS'21)

The main contribution of the paper is the following theorem:

Theorem: For each $k \geq 1$, reachability for $6k$ -VASSes is $\mathcal{F}_k$ -hard.

Using this theorem and properties of complexity classes $\mathcal{F}_k$, one concludes that VASS-reachability is Ackermann-hard.

In this lecture we will take a known $\mathcal{F}_k$ -hard problem and reduce to VASS-reachability.

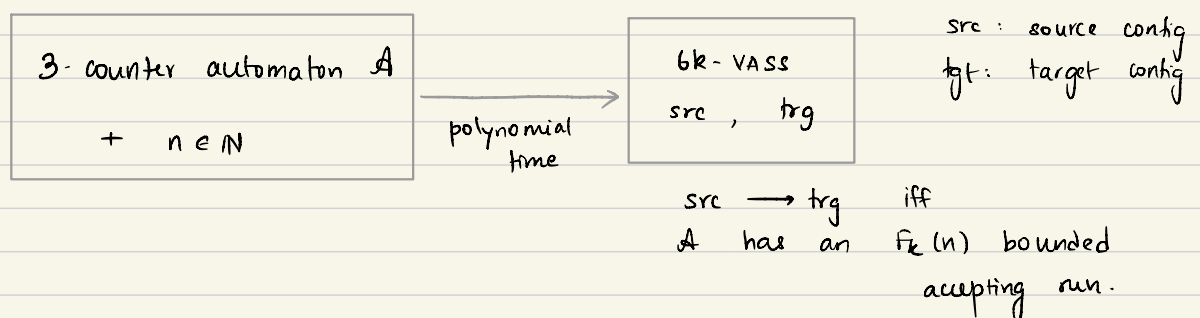
$\mathcal{F}_k$ -reachability for counter automata:

Input: Three-counter automaton $\mathcal{A}$, a number $n \in \mathbb{N}$

Output: Does $\mathcal{A}$ have an $F_k(n)$ -bounded accepting run?

A run is said to be N -bounded if all the counters are bounded by N .

Reduction:



Challenges in the reduction:

Challenge 1: Ensure that the counter values are bounded.

Challenge 2: How to simulate the zero-tests using a VASS

Plan for the rest of the lecture:

Part 1: Imposing boundedness of the counters

Part 2: Simulating the zero-tests

Part 3: Putting things together

Part 1: Imposing boundedness of the counters.

- Suppose (x, y, z) are the counters of $\mathcal{A}$ and suppose that we want to impose a bound of M on each of them.

- For each counter c we add an auxiliary counter $\bar{c}$

$$\begin{aligned} \therefore \quad x &\rightarrow \bar{x} \\ y &\rightarrow \bar{y} \\ z &\rightarrow \bar{z} \end{aligned}$$

- Maintain an invariant $c + \bar{c} = M$

Initially $c=0$; assume $\bar{c}=M$

$$\begin{array}{ccc} c += 1 \text{ in } \mathcal{A} & \longrightarrow & \begin{array}{l} c += 1 \text{ in VASS} \\ \bar{c} -= 1 \end{array} \end{array}$$

$$\begin{array}{ccc} c -= 1 \text{ in } \mathcal{A} & \longrightarrow & \begin{array}{l} c -= 1 \text{ in VASS} \\ \bar{c} += 1 \end{array} \end{array}$$

This transformation preserves the invariant.

Furthermore:

$$\text{when } c = M, \quad \bar{c} = 0$$

$\therefore c \neq 1$ will not be possible, since
it is coupled with $\bar{c} = 1$.

This invariant ensures that the counters are bounded by M .

So we have:

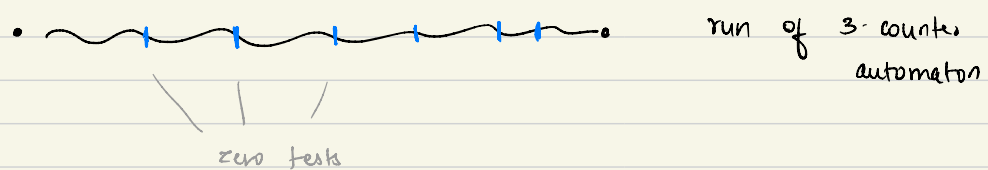
$$\underbrace{(x, y, z)}_{\text{original counters}}, \quad \underbrace{(\bar{x}, \bar{y}, \bar{z})}_{\text{their "duals"}}$$

Requirement: somehow start with $\bar{x} = \bar{y} = \bar{z} = M$
and $x = y = z = 0$

When $M = F_k(n)$ as will be required, we will introduce a
gadget that brings the dual counters to the
required value.

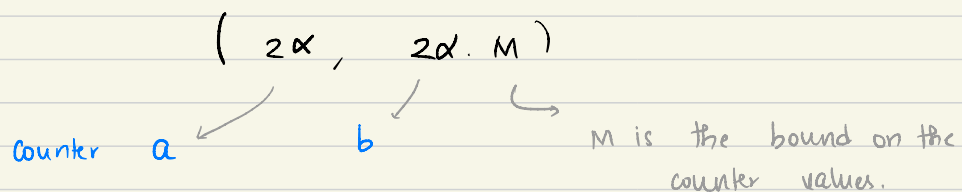
Part 2: Simulating the zero-tests

- In the previous lecture we saw how to simulate a fixed known number of zero tests using a controlling counter.
- Now we are looking for a run of the 3-counter machine which could have an arbitrary number of zero tests.
- The idea is to **guess** the number of zero tests and then simulate them in some way. The controlling counter technique is not applicable.

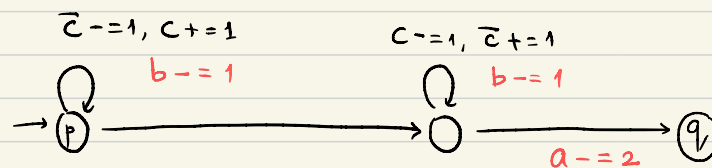


→ Guess that there are ' α ' zero tests

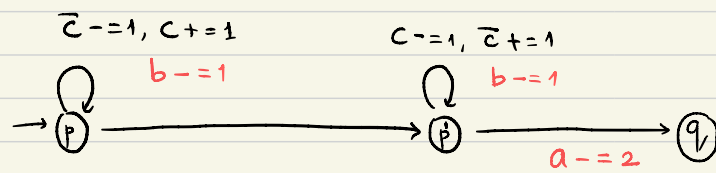
→ Have two special counters which somehow have initial values:



- For a zero test ' $c=0$ ', do the following.



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- From $p \rightarrow q$, counter 'a' is reduced by 2
- In p contents of $\bar{c}$ are transferred to c .
It can loop at most M times.
- In p' contents of c are transferred to $\bar{c}$. This can loop at most M times.
- Each transition reduces 'b' by 1.

Main observation: value of 'b' can reduce by $2M$ iff
 $c=0$ at the start of ' p '.

$\therefore$ In the tgt configuration of the VASS, we ask for $a=b=0$.

$\rightarrow$ There will be a run in the VASS only if the initial

guess $(2\alpha, 2\alpha \cdot M)$ is correct, and

$\rightarrow$ each zero-test of the counter automaton has been

faithfully simulated: i.e., there should be M loops on ' p '
and M loops on ' p' '.

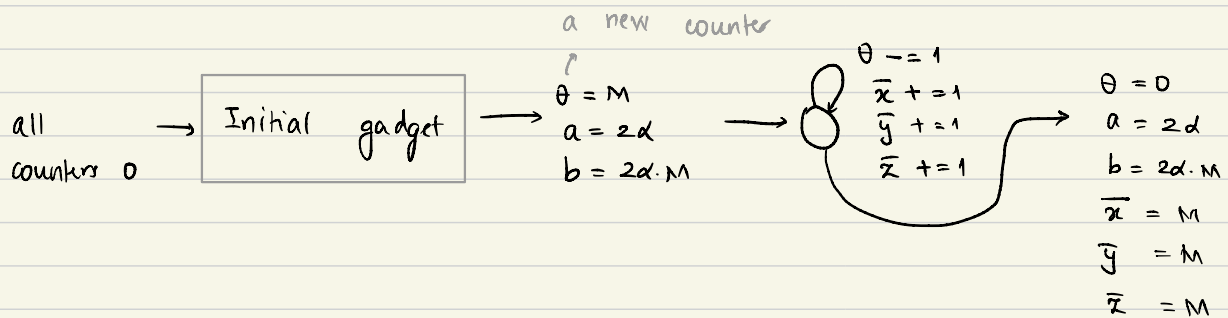
Part 3: Putting things together

From the previous two parts, we require:

→ 3 extra counters $\bar{n}, \bar{y}, \bar{z}$

→ 2 counters a, b

The goal is to construct an initial gadget as follows:



In the end we have the required values to simulate the
3-counter machine.

What is the M ? For F_k -reachability, $M = F_k(n)$.

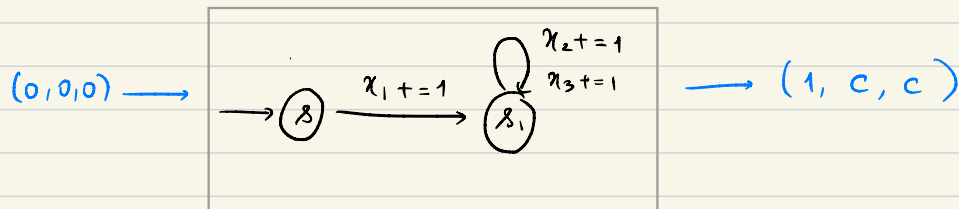
→ Therefore we need an initial gadget that can
produce large values for θ and 'b' and also guess the
number of zero tests α .

→ In this lecture we will show how to do this for F_2

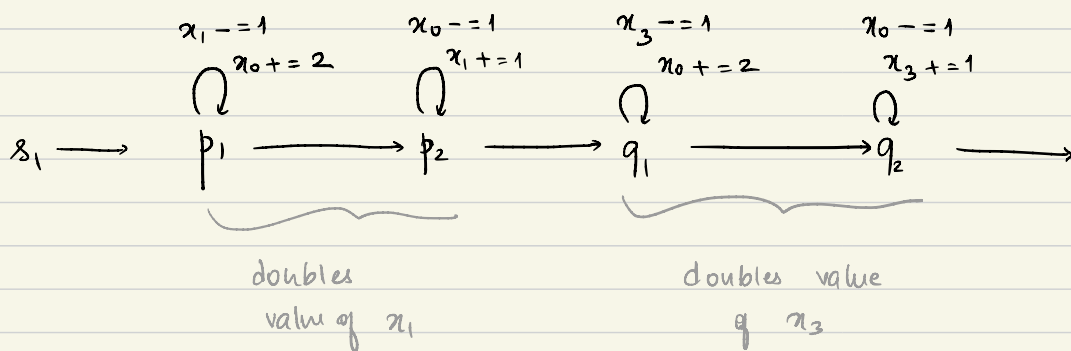
the paper explains it for F_k .

Getting to $(2^n, c, 2^n \cdot c)$:

- To show F_2 -hardness, given n , we require $M = 2^n$
- Let c be 2^d
- $\therefore$ We need a gadget to attain $(2^n, c, 2^n \cdot c)$
- The idea is to use controlling counter to enforce this.
- Let (x_1, x_2, x_3) be the counter.
- Initially all of them are 0.



- Now we need to double x_1 and x_3 'n' times. We use an auxiliary counter x_0 for this.



If we do this 'n' times, we get $(2^n, c, 2^n \cdot c)$ as needed.

→ To enforce that every loop is taken "exhaustively", we can use one additional controlling counter

↳ details of this are left as an exercise.

Summary:

- We have seen some seemingly "simple" looking techniques that enable showing the required hardness.
- In the paper, the authors mention that understanding hardness for low dimensions still remains widely open.