

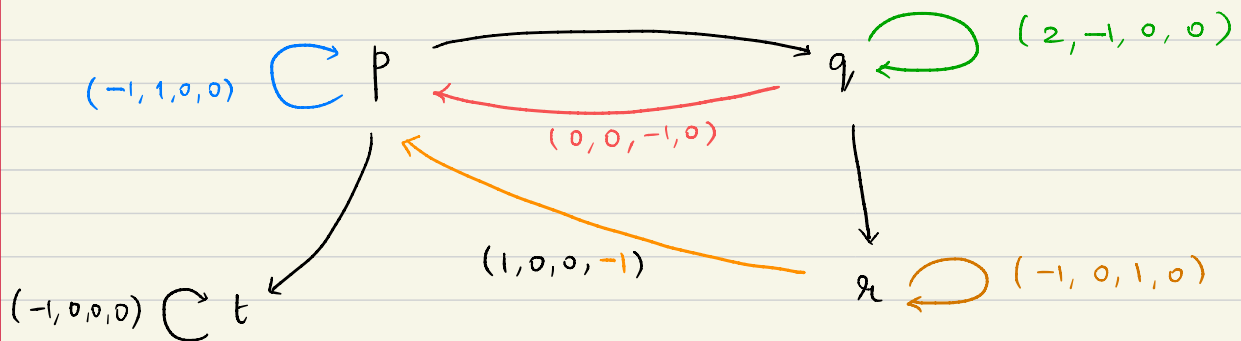
VASS REACHABILITY

Lecture 2: Some techniques to show lower bounds

Reference: Involved VASS Zoo, by Wojciech Czerwinski

↳ CONCUR'22 Invited Talk.

- In the previous lecture, we saw some examples that had long paths.
- for the reachability problem, we are given a source configuration (s), and a target configuration (t). To show hardness, we need examples s.t. 's' and 't' are small and the only paths between 's' and 't' are long. This may not always happen. For example, consider a variation of the "tower" example:



Source config: $p(1, 0, 1, n)$

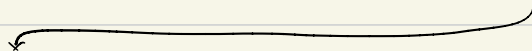
Target config: $t(0, 0, 1, 0)$

From previous lecture we know there is a path:

$$p(1, 0, 1, n) \longrightarrow p(\text{Tower}(n), 0, 1, 0) \longrightarrow t(\text{Tower}(n), 0, 1, 0) \\ \downarrow \\ t(0, 0, 1, 0)$$

However there is a short path:

$$p(1, 0, 1, n) \rightarrow q(1, 0, 1, n) \rightarrow r(1, 0, 1, n)$$



$$p(2, 0, 1, n-1) \rightarrow q(2, 0, 1, n-1) \dots$$

$\vdots$

$$p(n, 0, 1, 0) \rightarrow t(n, 0, 1, 0) \rightarrow t(0, 0, 1, 0).$$

- For hardness, we need some techniques/gadgets to **enforce** long paths between configurations.

In this lecture, we will see three different techniques that are helpful towards proving lower bounds.

The treatment will be focused towards providing an intuition, rather than description of all formal details.

↳ as in the reference mentioned.

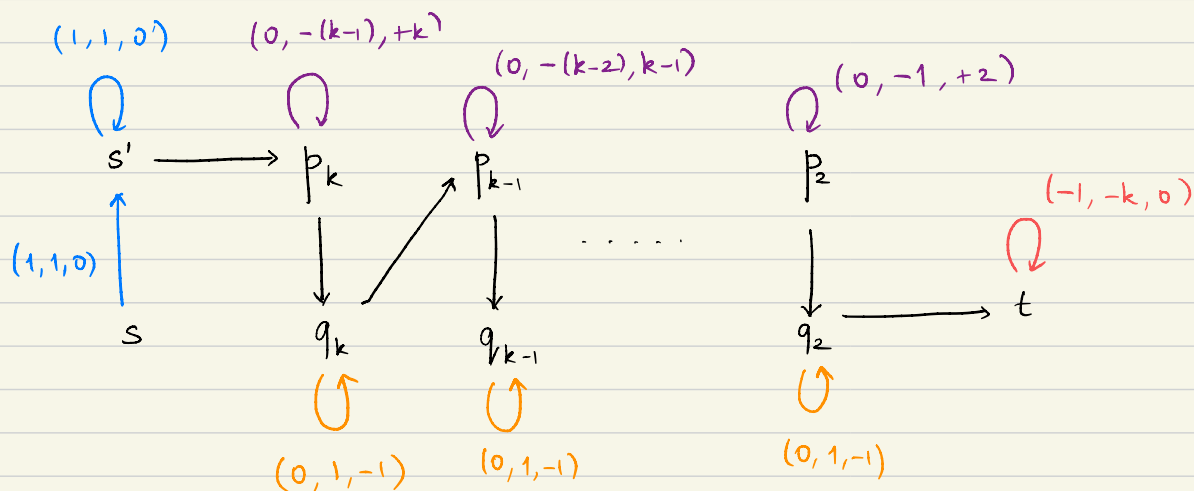
Part 1: Telescopic equations

Part 2: Controlling Counter

Part 3: Multiplication triples

Part 1: Teleportic equations

Consider the following gadget:



source configuration: $s(0, 0, 0)$

Target configuration: $t(0, 0, 0)$

call the counters $(\text{---}, \text{---}, \text{---})$
 $\begin{matrix} \uparrow & \uparrow & \uparrow \\ 1 & 2 & 3 \\ \text{first} & \text{second} & \text{third} \end{matrix}$

- Any run from $s(0, 0, 0)$ starts as follows:

$$s(0, 0, 0) \longrightarrow s'(1, 1, 0) \longrightarrow s'(2, 2, 0) \longrightarrow \dots \longrightarrow s'(N, N, 0) \longrightarrow p_k(N, N, 0)$$

- The first counter is not altered by the states $p_k, q_k, p_{k-1}, q_{k-1}, \dots, p_2, q_2$

Therefore at q_2 the value of the first counter is N .

- To make this counter 0 at t , the loop at t must be taken N times.

Therefore at q_2 the value of the second counter is Nk .

- Any run ends as follows:

$$q_2(N, Nk, 0) \longrightarrow t(N-1, (N-1)k, 0) \longrightarrow \dots \longrightarrow t(0, 0, 0)$$

- Hence the middle part of the run looks as follows:

$$p_k(N, N, 0) \longrightarrow \dots \longrightarrow q_2(N, Nk, 0)$$

- The second counter needs to be multiplied by k .

This is where the following telescopic equation is used:

$$k = \frac{k}{k-1} \cdot \frac{k-1}{k-2} \cdot \frac{k-2}{k-3} \cdots \frac{2}{1}$$

The gadget shown in the previous page implements the telescopic equation.

Remarks:

- q_i does not change the sum of second and third counter.
- p_i multiplies the sum by at most $\frac{i}{i-1}$

Recall that we start with sum N before p_k and in the end we have Nk .

By the telescopic equation above, we require each p_i to multiply the sum by exactly $\frac{i}{i-1}$.

The run therefore looks as follows:

$$p_k(N, N, 0) \rightarrow p_k(N, N-k-1, k) \rightarrow \cdots \rightarrow p_k(N, 0, \frac{N \cdot k}{k-1}) \rightarrow q_k(N, \frac{Nk}{k-1}, 0)$$

$$p_{k-1}(N, \frac{Nk}{k-1}, 0) \rightarrow \cdots \rightarrow p_{k-1}(N, 0, \frac{Nk}{k-2}) \rightarrow q_{k-1}(N, \frac{Nk}{k-2}, 0)$$

$$p_2(N, \frac{Nk}{2}, 0) \rightarrow \cdots \rightarrow p_2(N, 0, Nk) \rightarrow q_2(N, Nk, 0)$$

- Finally, notice that the run goes via $\frac{Nk}{i}$ for $i \in \{2, \dots, k-1\}$

Hence Nk needs to be divisible by $\text{lcm}(2, 3, \dots, k-1)$.

Czerwinski et al. (CONCUR'20) show that N needs to be exponential in k .

Summary

- The gadget implementing telescopic equation is a 3-VASS and it ensures that the shortest path from $s(0,0,0)$ and $t(0,0,0)$ requires an exponential length.
- Czerwiński et al. (CONCUR'20) use a more involved extension of this idea to provide a 4-VASS with shortest paths of doubly exponential length.

However VASS does not have the facility to zero-test

→ VASS + zero tests makes it a counter machine where reachability is undecidable with just 2 counters.

→ The question is how can we simulate "some" of the zero-tests within a VASS itself.

→ Firstly notice that in the example of the previous page, the test $c_4 = 0$ is unnecessary.

This is because the final configuration has $c_4 = 0$.

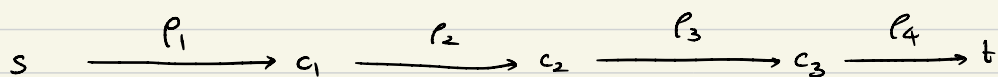
So we can use the final configuration to test for a counter value to be 0.

↳ this is sort of a "weak" zero-test.

→ We will now describe the idea of a controlling counter, which builds on this observation.

This idea was used in Czerwinski & Orlowski '21 in the Ackermann-hardness.

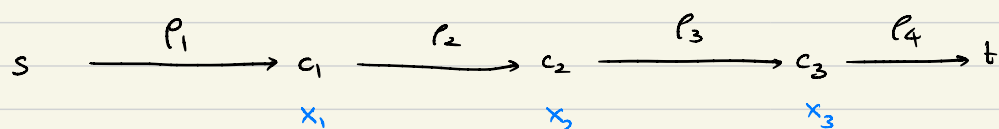
Consider a d-VASS V which has a counter x that we want to test for 0 in 3 moments during a run.



c_1, c_2, c_3 are configurations which can be specifically identified, let us say.

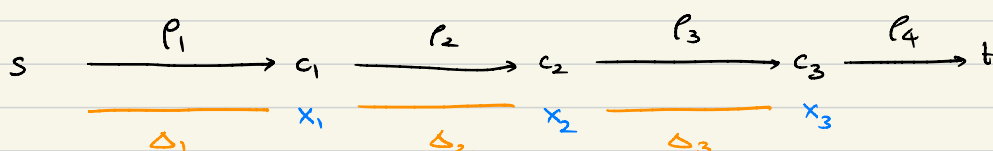
→ Assume we want to test $x=0$ after c_1, c_2 and c_3 .
Assume $x=0$ at s .

→ Let x_1, x_2, x_3 be the value of x at c_1, c_2, c_3 respectively.



$$x_1 = x_2 = x_3 = 0 \quad \text{iff} \quad x_1 + x_2 + x_3 = 0 \quad (\text{as counter values are non-negative})$$

- Let $\Delta_1, \Delta_2, \Delta_3$ be the effect of the run on counter x in p_1, p_2, p_3 respectively.



$$\begin{aligned} \therefore x_1 &= \Delta_1 \\ x_2 &= \Delta_1 + \Delta_2 \\ x_3 &= \Delta_1 + \Delta_2 + \Delta_3 \end{aligned}$$

$$x_1 + x_2 + x_3 = 3\Delta_1 + 2\Delta_2 + \Delta_3$$

→ Therefore we can add a special "controlling-counter" y

→ Initially, $y=0$

→ At p_1 , each update to x by C is matched by an update $3C$ to y .

→ At p_2 , updating x by C is matched with $2C$ to y .

→ At p_3 , update to x is simply copied to y too.

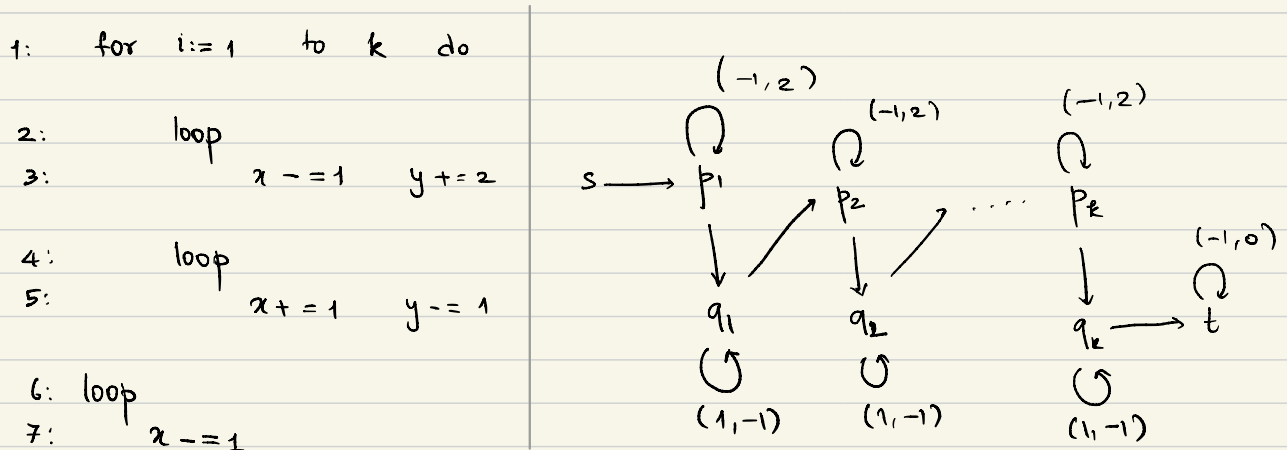
∴ At the end of the run, value of y would be: $3\Delta_1 + 2\Delta_2 + \Delta_3$

∴ Asking for $y=0$ in the final configuration would entail

$x=0$ in the three intermediate configurations c_1, c_2, c_3 .

Example: Here's an application of the controlling counter technique to a 2-VASS.

A VASS can also be described as a counter program.



Suppose that initial values are $x=1, y=0$

- If all loops are maximally taken we reach $q_k(2^k, 0)$ and the loop in t can be fired 2^k times to reach $(0, 0)$.
- However there is a short run that avoids all the intermediate loops.

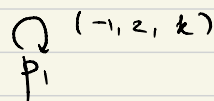
From $p_i \rightarrow q_i$ we want $x=0$.

From $q_i \rightarrow p_i$ we want $y=0$.

→ Using a controlling counter, we could instrument the VASS as follows.

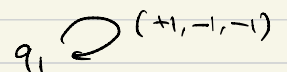
* At p_1 : x has k -more zero tests and y has k more zero tests.

∴ x has to be updated with: $k \cdot (-1) + k(2) = k$



At q_1 : x has $(k-1)$ more zero-tests, y has k more:

∴ z 's update: $(k-1)(+1) + k \cdot (-1) = -1$



* At p_i : x is waiting for $k-i+1$ zero tests, y is waiting for $k-i+1$ test too

$$\begin{aligned} \therefore Z's \text{ update: } & (k-i+1)(-1) + (k-i+1)(2) \\ & = -k + i - 1 + 2k - 2i + 2 \\ & = k - i + 1 \end{aligned}$$

$$p_i \ni (-1, 2, k-i+1)$$

At q_i : x is waiting for $k-i$ zero tests, while y is waiting for $k-i+1$ zero tests.

$$\therefore Z's \text{ update: } (k-i) \cdot (1) + (k-i+1)(-1) = -1$$

$$q_i \ni (1, -1, -1)$$

The new counter program would be:

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1: for  $i := 1$  to  $k$  do
2:   loop
3:      $x -= 1$     $y += 2$     $z += k-i+1$ 
4:   loop
5:      $x += 1$     $y -= 1$     $z -= 1$ 
6: loop
7:    $x -= 1$ 

```

Summary:

- We have seen two techniques in this lecture.
- Telescopic equation technique has been used to show a doubly exponential length shortest path in a 4-VASS
- Controlling-counter technique has been used in the Ackermann-hardness proof.