

VASS REACHABILITY

Lecture 1: Introduction to VASS through examples

Reference: Involved VASS Zoo, by Wojciech Czerwinski

↳ CONCUR'22 Invited Talk.

Part 1: Example 1 - exponential state space

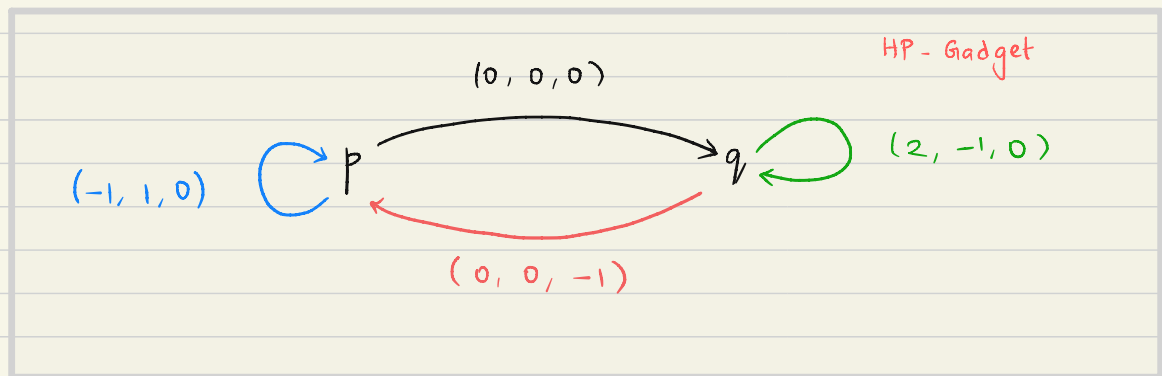
Part 2: VASS reachability, fast growing functions, short history of results

Part 3: Example 2 - doubly exponential state space

Part 4: Example 3 - tower of exponentials state space

Part 5: Beyond tower

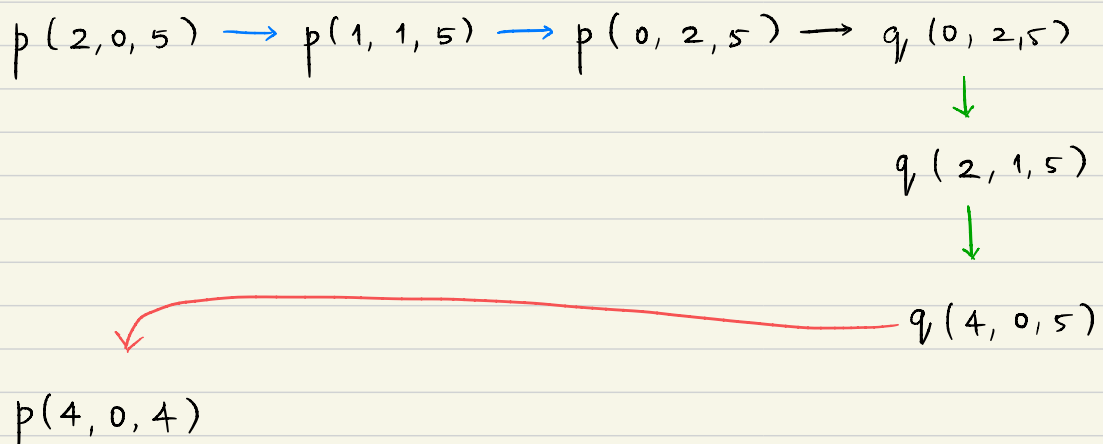
Part 1: Example 1



A VASS - introduced in Hoppercroft-Pansiot '78

VASS is given by :

- a finite set of states
 - a finite set of counters - the no. of counters is called its "dimension"
 - edges: contain a vector of integers, with size of vector equal to VASS' dimension
- Counters cannot go below 0.
 - Here is an example of a behaviour of the HP-gadget.
 - Suppose we start with initial "vector" $(2, 0, 5)$



- In one cycle: third counter reduces by 1, first counter is doubled.

If we start initially with $(l, 0, n)$

$$p(l, 0, n)$$

$\downarrow$

$$p(2l, 0, n-1)$$

$\downarrow$

$$p(2^2 l, 0, n-2)$$

$\downarrow$

$$p(2^n l, 0, 0)$$

→ we keep doubling n times resulting in $2^n l$.

Intuitively:

→ Blue transition transfers value from Counter 1 to Counter 2.

→ Green transition value of counter 2 is transferred to counter 1 along with multiplication by 2.

→ The red transition reduces the third counter by 1 and governs the number of times the cycle $p \rightarrow q \rightarrow p$ can be taken.

$$\therefore p(l, 0, n) \rightarrow p(2l, 0, n-1) \rightarrow p(2^2 l, 0, n-2) \rightarrow \dots \rightarrow p(2^n l, 0, 0)$$

Remarks about the HP-gadget:

→ No. of configurations reachable from an initial vector is **finite**.

→ Size of this state space is at least exponential.

$$p(l, 0, n) \rightarrow p(x, y, n) \quad \forall x+y=n$$

→ can be shown that the reachable state space is almost exponential

Part 2: VASS reachability, fast-growing functions, a short history of results.

Reachability problem:

Given a VASS, and two configurations 's' and 't'.

Does there exist a run of the VASS from 's' to 't'?

Main result:

VASS - reachability is Ackermann-complete.

What is Ackermann-completeness? We will now see a hierarchy of fast growing functions.

fast growing functions:

$$F_i : \mathbb{N} \rightarrow \mathbb{N} \quad i \geq 1$$

$$\begin{aligned} F_1(n) &= 2n \\ F_{k+1}(n) &= \underbrace{F_k \circ F_k \circ \dots \circ F_k}_n(1) \\ &\quad \text{n times} \end{aligned}$$

$$F_2(n) = \underbrace{F_1 \circ F_1 \circ \dots \circ F_1}_n(1) = 2^n$$

$$F_3(n) = \underbrace{F_2 \circ F_2 \circ \dots \circ F_2}_n(1) = \overset{2}{\nearrow} \underset{n}{\nwarrow} 2^{2^{2^{\dots}}} = \text{Tower}(n)$$

What is $F_4(n)$?

$$F_4(1) = F_3(1) = F_2(1) = F_1(1) = 2$$

$$\begin{aligned} F_4(2) &= F_3 \circ F_3(1) \\ &= F_3(2) \\ &= 2^2 \end{aligned}$$

$$\begin{aligned} F_4(3) &= F_3 \circ F_3 \circ F_3(1) \\ &= F_3 \circ F_3(2) \\ &= F_3(2^2) \\ &= \underset{\substack{\nearrow \\ \text{Tower of height } 2^2}}{2^{2^2 \dots^2}} \end{aligned}$$

$$\begin{aligned} F_4(4) &= F_3 \circ F_3 \circ F_3 \circ F_3(1) \\ &= F_3 \circ F_3 \circ F_3(2) \\ &= F_3 \circ F_3(2^2) \\ &= F_3(\text{Tower}(2^2)) \\ &= \text{Tower of height Tower}(2^2) \end{aligned}$$

...

Ackermann- function:

$$\text{Ack}(n) = F_\omega(n) = F_n(n)$$

↑
compose F_n n times

$$\begin{aligned} F_1(1) &= 2 & F_2(2) &= F_2 \circ F_2(2) = 2^2 & F_3(3) &= F_3 \circ F_3 \circ F_3(3) \\ & & & & &= \text{Tower}(\text{Tower}(\text{Tower}(3))) \end{aligned}$$

COMPLEXITY CLASSES BASED ON $\mathcal{F}_i$

Roughly:

$\mathcal{F}_i$ is the class of decision problems that can be solved by a deterministic Turing machine in time:

$$\mathcal{F}_i(p(n))$$

where:

- n is size of input
- $p: \mathbb{N} \rightarrow \mathbb{N}$ is a function computable in deterministic time $\mathcal{F}_{i-1}^m(n)$ for some $m \in \mathbb{N}$.
↓
 m^{th} power of $\mathcal{F}_{i-1}$

$\mathcal{F}_i$ -hardness:

A problem P is $\mathcal{F}_i$ -hard if every problem in $\mathcal{F}_i$ can be reduced to P by a reduction computable in time of lower order $\mathcal{F}_{i-1}^m$ for some $m \in \mathbb{N}_1$.

To show Ackermann-hardness, the following result has been proved by

Czerwiński & Orlikowski (FOCS'21)

Reachability problem for $(2k+4)$ -VASSes is $\mathcal{F}_k$ -hard

Short history of the problem:

Lipton '76 : EXPSPACE - hardness for coverability:

↓
given s, t is there a path from s to t' for some $t' \geq t$.

Rackoff '78 : coverability in EXPSPACE

Mayr '81 : decidability of reachability

Kosaraju '82, Lambert '92: simplifications

Blondin et al '15: reachability PSPACE-complete for 2-VASSes

Leroux - Schmitz '15: reachability in cubic-Ackermann

Leroux - Schmitz '19: reachability in Ackermann

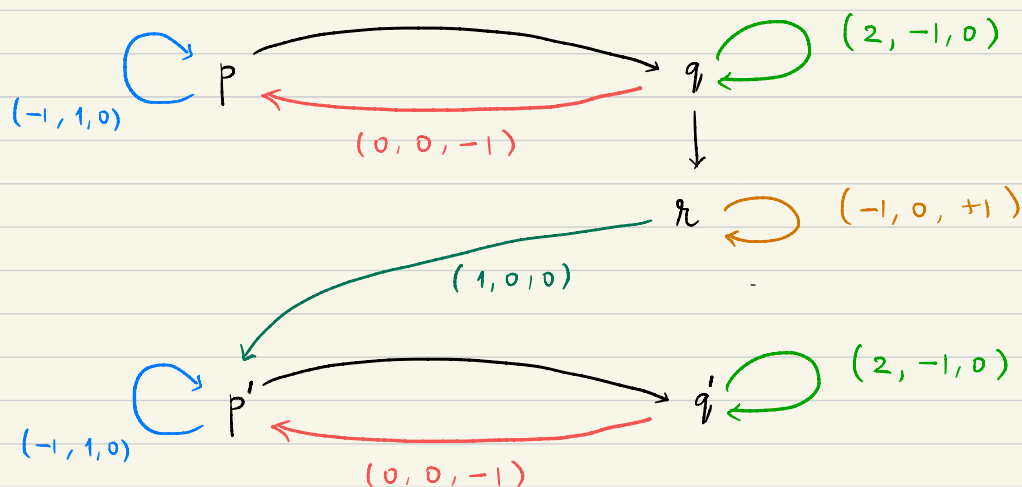
Czerwiński, Lasota, Łazic, Leroux, Mazowiecki '19: Tower-hardness

Leroux '21, Czerwiński & Orlikowski '21: Ackermann-hardness

Open problems:

→ check Wojciech's webpage for a list of current open problems.

Part 3: Example 2 - doubly exponential state-space.



- p, q form an HP-gadget
- so starting from $(1, 0, n)$ we have:

$$p(1, 0, n) \longrightarrow p(2^n, 0, 0) \longrightarrow q(2^n, 0, 0)$$

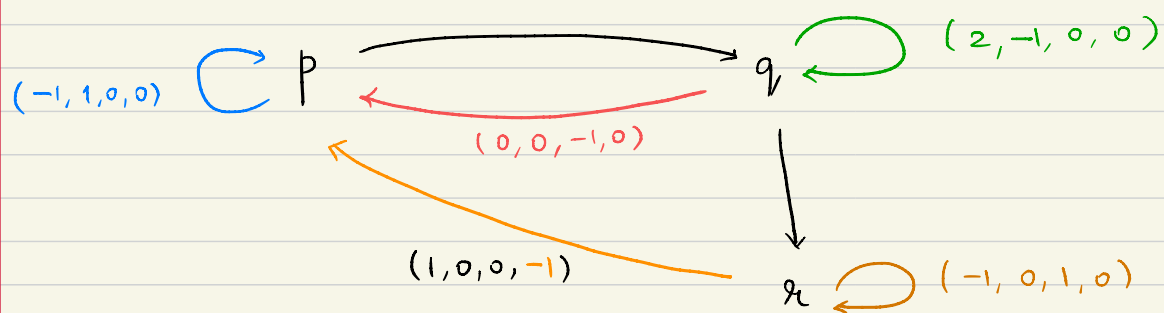
- state 'r' transfers value from counter 1 to 3. Then we get:

$$q(2^n, 0, 0) \longrightarrow r(2^n, 0, 0) \longrightarrow r(0, 0, 2^n) \longrightarrow p(1, 0, 2^n)$$

- The bottom HP-gadget then produces:

$$p(1, 0, 2^n) \longrightarrow p(2^{2^n}, 0, 0)$$

Part 4: Example 3 - tower state space.



- p, q , form an HP-gadget. suppose we start with $(1, 0, 1, n)$

$$p(1, 0, 1, n) \longrightarrow p(2, 0, 0, n) \longrightarrow q(2, 0, 0, n)$$

- Then r transfers values from counter 1 to 3 and then the transition back to p reduces 1 from the last counter.

$$q(2, 0, 0, n) \longrightarrow r(2, 0, 0, n) \longrightarrow r(0, 0, 2, n)$$

$$p(1, 0, 2, n-1)$$

$$\therefore p(1, 0, 1, n) \longrightarrow p(1, 0, 2, n-1)$$

Continuing, we get:

$$p(1, 0, 1, n) \longrightarrow p(1, 0, 2, n-1) \longrightarrow p(1, 0, 2^2, n-2)$$

↓

$$p(1, 0, 2^2, n-3)$$

↓

$$p(1, 0, 2^{n-2}, 0)$$

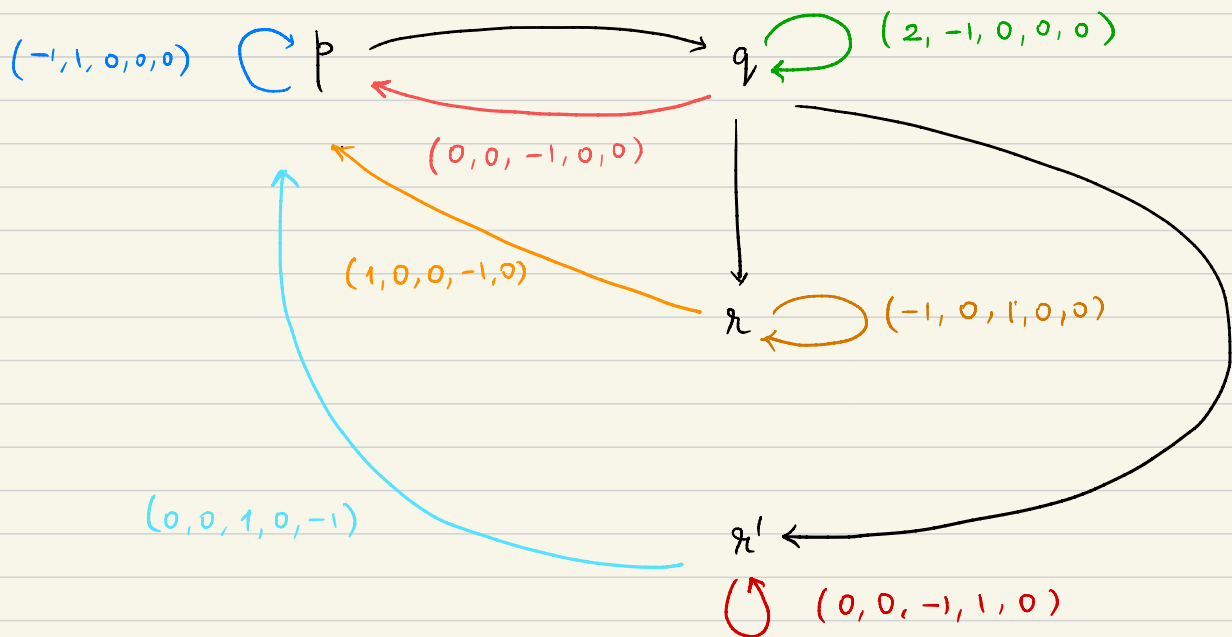
$\therefore$ The VASS shown in the previous page has 4 counters and builds a run whose counter value attains $F_3(n)$.

$$\begin{array}{c} p(1, 0, 1, n) \\ \Downarrow \\ p(1, 0, 2, n-1) \\ \Downarrow \\ p(1, 0, 2^2, n-2) \\ \Downarrow \\ \vdots \\ p(1, 0, F_3(n), 0) \end{array}$$

Remark:

Notice that the reachable state space from any initial configuration is finite for this VASS.

Part 5: Beyond Tower



$$p(1, 0, 1, 1, n) \longrightarrow p(1, 0, F_3(1), 0, n) \longrightarrow r'(1, 0, F_3(1), 0, n)$$

↓

$$r'(1, 0, 0, F_3(1), n)$$

↖

$$p(1, 0, 1, F_3(1), n-1) \longrightarrow p(1, 0, F_3^2(1), 0, n-1) \longrightarrow r'(1, 0, F_3^2(1), 0, n-1)$$

↓

$$r'(1, 0, 0, F_3^2(1), n-1)$$

↖

$$p(1, 0, 1, F_3^2(1), n-2)$$

⋮

$$p(1, 0, 1, F_3^3(1), n-3)$$

⋮

$$p(1, 0, 1, F_3^n(1), 0) = p(1, 0, 1, F_4(1), 0)$$

→ Using 5 counters, we could manufacture a run attaining value $F_4(1)$.

Exercise: Design a $(d+1)$ -VASS with $O(d)$ states and counter update in $\{-1, 0, +1, 2\}$ and produce an initial vector which contains in its reachable state space a vector with a counter value $F_d(1)$.

Summary:

- Intro to VASS
- Reachability problem
- Examples to show that simple VASSes can have complicated behaviours
 - ↳ can be extended to give d-VASSes with finite reachability sets of size $F_{d-1}(n)$ where n is the size of the source configuration.

Upper bound:

Leroux & Schmitz [LICS'19] show that:

- Reachability in d-VASS can be solved in F_{d+4} complexity

Lower bound:

Czerwiński & Orlikowski [FOCS'21] show that:

- Reachability in d-VASS is $F_{(d-4)/2}$ -hard.

There is still a gap. Several improvements have happened since then, esp. with hardness.