HISTORY DETERMINISM

Lecture 14: Summary of the HD module

Part 1: Expressiveness

Part 2: Succinctness

Part 3: characterization

Part 4: Language inclusion

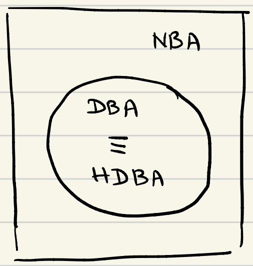
Part 1: Expressiveness

Co Büchi

Non-deterministic co Büchi
$\equiv$
deterministic co Büchi

So HD does not add expressivity compared to determinism

Büchi



Every HD BA can be converted to a DBA with at most n^2 blow-up.

For parity, non-determinism coincides with determinism. So HD adds no power compared to determinism.

Part 2: Succinctness

coBüchi : HD coBüchi automata are exponentially more succinct than deterministic coBüchi automata.

Example from Kuperberg & Skrzypczak ICALP'15

- Alphabet $\Sigma = \{ \iota, \sigma, \pi, \# \}$ 4 symbols.

- These symbols stand for permutations over the set $\{0, 1, 2, \ldots, 2n-1\}$

ι : identity permutation

$\#$: identity permutation on $\{1, 2, \ldots, 2n-1\}$ and undefined on 0.

σ : $k \mapsto (k+1) \bmod 2n$

π : $0 \mapsto 1$
$1 \mapsto 0$
$\left.\begin{array}{l} 2 \to 2 \\ \vdots \\ 2n-1 \to 2n-1 \end{array}\right\}$ identity

Figure from the paper in the next page.

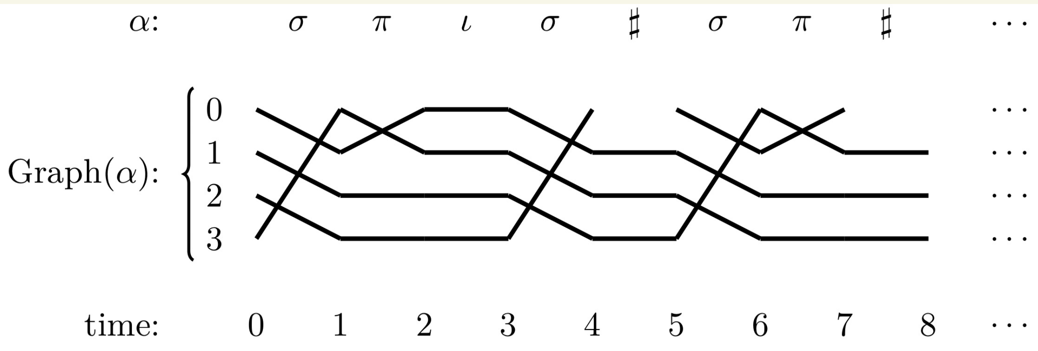


Fig. 1. The infinite sequence of relations on the set $\{0, \ldots, 3\}$ (i.e. $n = 2$) represented by an ω-word $\alpha \in A^\omega$.

$L_n = \{ \alpha \in \Sigma^\omega \mid \text{Graph}(\alpha) \text{ contains an infinite path} \}$

Exercise: Take $n=2$. Give an example of a word not in L_2.

HD coBüchi automaton C_n for L_n:

$Q = \{ \perp, 0, 1, 2, \ldots, 2n-1 \}$

- The states $\{0, 1, 2, \ldots, 2n-1\}$ are deterministic.

  Reading $a \in \Sigma$ from a state q moves to the successive state given by the permutation a.

  → or to $\perp$ if $a = \#$ and $q = 0$.

- The state $\perp$ is non-deterministic - on $a \in \Sigma$, the automaton can move to any state $q' \in \{0, 1, \ldots, 2n-1\}$.

Initial state: $\perp$

Rejecting transitions : $\perp \xrightarrow{a} q'$

Priority $\underline{1}$

Others have priority 2

So eventually $\perp$ should not be visited.

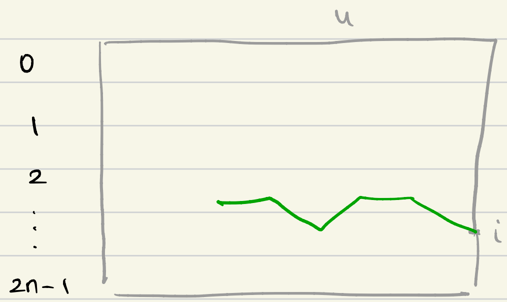
$\Rightarrow$ there is an infinite path in Graph(α).

Why is C_n HD?

Strategy for Eve in letter game.

→ Consider a finite word $u \in \Sigma^*$

→ Look at Graph(u)



For each i, there is a "path" starting from some earlier point and ending at i after reading u.

On 'u' if the automaton is at state $\perp$,

the strategy chooses an 'i' s.t. it has the longest path at u.

→ For an ω-word $\alpha \in L_n$, Graph(α) contains an infinite path. Therefore the above strategy would ensure that eventually an infinite path would be chosen and $\perp$ will never be visited again

HD Büchi - succinctness:

↪ open problem

Result at the time of writing these notes (September '2026)

→ There is a language that can be accepted by an HD Büchi automaton with 65 states.

Every deterministic Büchi automaton for this language requires > 66 states.

— Casares, Prakash, Thejaswini - 2026.

Part 3: Characterization.

For Büchi & coBüchi, we have the following:

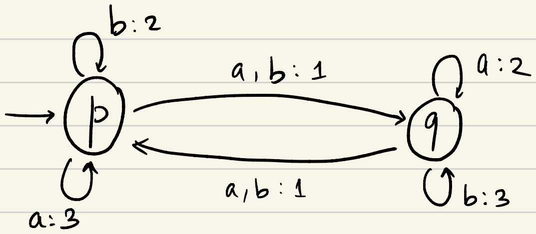
> $\mathcal{A}$ is HD iff
>
> Eve wins the 2-token game iff
>
> Eve wins the Joker game.

We have seen an elaborate proof of this characterization in our course.

Complexity: The above characterization also gives a PTIME complexity for deciding whether a given Büchi / coBüchi automaton is HD.

↳ simply build the arena for the Joker game or the 2-token game.

Joker games are not enough for parity automata:



Eve wins Joker game with this strategy:

→ If Adam's & Eve's tokens are at different states, then Eve switches states.

→ Else she continues in the same state.

→ if eventually both Eve and Adam remain in the same state then Adam's run satisfies parity condition iff Eve's run satisfies parity condition.

→ if Adam switches infinitely often, then priority 1 is seen infinitely often. So his token will be non-accepting and Eve wins.

→ Adam can play Joker without actually taking the shifting transition. But he can do this only finitely many times.

Hence we conclude that Eve wins plays conforming to the said strategy.

This automaton is not HD!

↳ Adam wins the letter game.

Firstly : every word is accepted by above automaton.

Adam's strategy in letter game:

→ When Eve is at 'p' play 'a'

When Eve is at 'q' play 'b'.

Therefore no transition of even priority is ever taken.

⇒ Automaton is not HD.

2-token theorem:

Let $\mathcal{A}$ be a parity automaton.

$\mathcal{A}$ is HD

$\iff$

Eve wins the 2-token game on $\mathcal{A}$.

↳ Proof of this theorem in Keya Prakash's thesis.

Part 4: Language inclusion

$\mathcal{A}, \mathcal{B}$ parity automata

is $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$?

$\rightarrow$ PSPACE-complete in general.

Now consider this problem:

$\mathcal{A}$: parity aut. $\mathcal{H}$: HD parity automaton

is $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{H})$?

Lemma: $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{H})$ iff $\mathcal{H}$ simulates $\mathcal{A}$.

Proof: If $\mathcal{H}$ simulates $\mathcal{A}$, then $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{H})$.

Suppose $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{H})$. Since $\mathcal{H}$ is HD, in the game $\mathrm{Sim}(\mathcal{H}, \mathcal{A})$, Eve can play her strategy in the letter game on $\mathcal{H}$.

Since $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{H})$, whenever Adam produces an accepting word, in $\mathcal{A}$, Eve's run will be accepting in $\mathcal{H}$.

$\therefore$ $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{H})$ iff $\underbrace{\text{Eve wins } \mathrm{Sim}(\mathcal{H}, \mathcal{A})}_{\text{deciding this is in NP.}}$

$\therefore$ We get an NP algorithm for $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{H})$.

Summary:

- → This course introduced HD nns and presented some of its ingredients.
- → Plenty of open problems and insights left to be explored.
    - → See Chapter 9 of Keya's thesis for a list of problems.