# HISTORY DETERMINISM

**Lecture 13:** Characterizing History-Determinism in Büchi automata - Part III

**Reference:** Section 5.2 of Keya Prakash's Ph.D. thesis

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In today's lecture, we will show that the rank-reduction procedure preserves the following invariants.

I1: Simulation-equivalence to the original automaton $\mathcal{A}$

I2: Eve winning the 1-token game from everywhere

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Recall the rank-reduction procedure.

$$\mathcal{A} = \mathcal{A}_0 \xrightarrow[\text{Step 2}]{} \mathcal{A}_0' \xrightarrow[\text{Step 3}]{} \mathcal{A}_1 \longrightarrow \mathcal{A}_1' \longrightarrow \mathcal{A}_2 \longrightarrow \cdots \mathcal{A}_i \longrightarrow \mathcal{A}_i' \longrightarrow \mathcal{A}_{i+1} \longrightarrow \cdots \mathcal{A}_N$$

Step 2 removes transitions $q \xrightarrow{a:1} q'$ s.t. $opt_i(q) < opt_i(q')$

Step 3 changes priorities of $q \xrightarrow{a:1} q'$ s.t. $opt_i(q) > opt_i(q')$ to 1.

To prove that invariants I1 and I2 are preserved throughout the computation, we will assume $\mathcal{A}_i$ satisfies the invariants, and Steps 2 & 3 preserve the invariants.

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**Part 1:** Invariants are preserved after Step 2

**Part 2:** Invariants are preserved after Step 3

**Part 3:** Stabilization

**Part 1:** Invariants are preserved after step 2.

$$\mathcal{A}_i \xrightarrow[\text{Step 2}]{} \mathcal{A}_i'$$

$\mathcal{A}_i'$ is a subautomaton with some transitions removed from $\mathcal{A}_i$.

- To show I1 it is sufficient to prove:

  Eve wins $G1(\mathcal{A}_i' : q_0 \,;\, \mathcal{A}_i : q_0)$

  This will entail that Eve wins the simulation game

  $\text{Sim}(\mathcal{A}_i' \,;\, \mathcal{A}_i)$

  Hence $\mathcal{A}_i'$ simulates $\mathcal{A}_i$

  As $\mathcal{A}_i'$ is a subautomaton of $\mathcal{A}_i$, $\mathcal{A}_i$ simulates $\mathcal{A}_i'$ trivially.

- To show I2 it is sufficient to prove:

  Eve wins $G1((\mathcal{A}_i' : q) \,;\, (\mathcal{A}_i : p))$ $\forall\, q, p$ that are weakly co-reachable.

  This would entail that Eve wins $G1(\mathcal{A}_i' : q \,;\, \mathcal{A}_i' : p)$.

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From the discussion above, we will show the following.

$$\boxed{\text{Eve wins } G1((\mathcal{A}_i', q), (\mathcal{A}_i, p)) \text{ for all } q, p \text{ that are WCR}}$$

**Proof idea:**

- We will consider the game $\mathcal{G}_1(\mathcal{A}_i)$. Eve wins from every vertex.

- Using the strategy in $\mathcal{G}_1(\mathcal{A}_i)$ we will develop a strategy for Eve in $G1((\mathcal{A}_i', q), (\mathcal{A}_i, p))$

- Let $\tau$ be an 'optimal' positional winning strategy for Eve in $\mathcal{G}_1(\mathcal{A}_i)$

- $\tau$ satisfies the remarks listed in the end of Part 1 of Lecture 12, and the properties about ranks in Part 2 of Lecture 12.

- The main issue is about what happens when $\tau$ gives an edge that has been deleted.

$$
\begin{array}{ccc}
\text{Eve} & & \text{Adam} \\
q & & p \\
\downarrow & \downarrow a_1 & \downarrow \\
q_1 = \tau(q, a_1, p) & & p_1 \\
\downarrow & \downarrow a_2 & \downarrow \\
q_2 = \tau(q_1, a_2, p_1) & & p_2 \\
& \vdots &
\end{array}
$$

Problem: What happens when $\tau(q_j, a_{j+1}, p_j)$ has been deleted in $\mathcal{A}_i'$.

- To deal with this situation, Eve maintains a memory token.

Initial configuration:

| Eve's token | Adam's token | Eve's memory token |
|---|---|---|
| $q_0 = q$ | $p_0 = p$ | $r_0$ |

$r_0 \in \mathrm{WCR}(\mathcal{A}_i, q_0)$ s.t.

$\mathrm{opt}_i(q_0) = \mathrm{rank}(q_0, r_0)$

$r_0$ corresponds to the state which witnesses the optimal rank for $q_0$.

Strategy: After $j$ rounds assume the following configuration.

| Eve's token | Adam's token | Eve's memory token |
|---|---|---|
| $q_j$ | $p_j$ | $r_j$ |

$q_j, p_j, r_j$ are all weakly co-reachable.

→ Suppose Adam chooses letter $a_j$

→ Eve plays from $q_j$ imagining Adam's token is at $r_j$.

Suppose $\boxed{\tau(q_j, a_j, r_j) = \delta_j := q_j \xrightarrow{a_j : c_j} q_{j+1}}$

## Case 1:

$\delta_j$ is a 0-transition

- Eve takes this transition to go to $q_{j+1}$ on her token
- Eve resets memory token to $r_{j+1} \in WCR(\mathcal{A}_i, q_{j+1})$

  s.t.

  $rank(q_{j+1}, r_{j+1}) = opt_i(q_{j+1})$

## Case 2:

$\delta_j$ is a 1-transition that has not been removed yet

- Eve moves token to $q_{j+1}$
- Eve updates memory to

  $r_{j+1} = \tau(r_j, a_j, p_j)$

## Case 3:

$\delta_j$ is no-longer a transition in $\mathcal{A}_i'$

- Eve finds an $s_j$ s.t.

  $opt_i(q_j) = rank(q_j, s_j)$

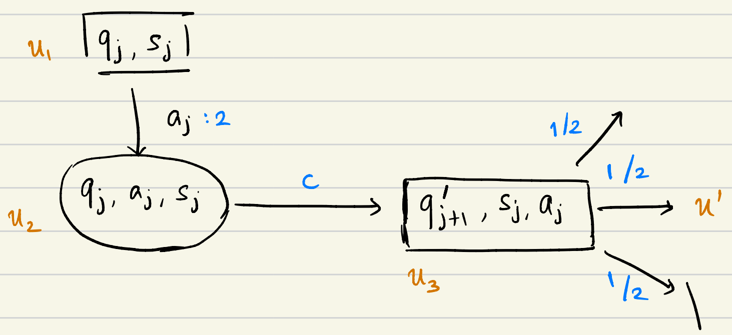
  $\delta' := \tau(q_j, a_j, s_j) = q_j \xrightarrow{a_j} q'_{j+1}$

  Claim 1: $\delta'$ exists in $\mathcal{A}_i'$

- Eve moves to $q'_{j+1}$
- Eve updates memory to

  $r_{j+1} = \tau(s_j, a_j, p_j)$

Proof of claim 1: We have $opt_i(q_j) = rank(q_j, s_j)$




Firstly: $rank(u_1) \geq rank(u_2)$

Recall from definition of $\mathcal{G}_i$ that

$c$ is either 0 or 2.

$\therefore$ $rank(u_2) \geq rank(u_3)$.

Finally, since Adam's outgoing transitions from $u_3$ have priority 1 or 2.
$rank(u_3) \geq rank(u')$ for all successors $u'$.

$\therefore \quad rank(q_j, s_j) \geq rank(q'_{j+1}, s_{j+1})$

$\hookrightarrow$ $\forall$ successors $s_{j+1}$.

$\therefore \quad opt_i(q_j) \geq opt_i(q'_{j+1})$

$\Rightarrow \quad q_j \longrightarrow q'_{j+1}$ is not deleted.

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Now we need to show that the strategy that has been proposed is winning for Eve in

$$G1((A'_i, q), (A_i, p))$$

- Call this newly proposed strategy $\tau'$.

A play of $G1((A'_i, q), (A_i, p))$ according to $\tau'$ would look as follows:

$$\begin{array}{c}(q_0, p_0, r_0)\\ \downarrow a_0\\ (q_1, p_1, r_1)\\ \downarrow a_1\\ (q_2, p_2, r_2)\\ \downarrow\\ \vdots\end{array}$$

Here are some properties about such a play:

Each move falls under either Case 1, Case 2 or Case 3.

We

Case 3:

Let us closely look at a case 3 more.

Starting position: $(q_{V_j}, p_j, r_j)$

Adam plays: $a_j$

Eve looks at the game $\mathcal{G}_1(\mathcal{A}_i)$ and the vertex:

$$\begin{array}{ccc} (q_j, a_j, r_j) & \longrightarrow & \text{rank} = x \\ 2 \longrightarrow \big\downarrow \tau & & \\ (q_{j+1}, r_j, a_j) & \longrightarrow & \text{rank} = y \end{array}$$

But $q_j \xrightarrow{a_j} q_{j+1}$ has been deleted because

$$\boxed{opt_i(q_j) < opt_i(q_{j+1})} \longrightarrow (1)$$

Suppose $rank(q_j, a_j, r_j) = x$, $rank(q_{j+1}, r_j, a_j) = y$

Since $q_j \xrightarrow{a_j} q_{j+1}$ has been deleted, its priority in $\mathcal{A}_i$ is 1,

hence priority of above transition in $\mathcal{G}_1(\mathcal{A}_i)$ is 2 (marked in b/w above)

By monotonicity of ranks: $\boxed{x \geq y} \longrightarrow (2)$

Now:

$$opt_i(q_j) \leq x$$

$$opt_i(q_{j+1}) \leq y$$

If $opt_i(q_j) = x$, then from (1) we get $x < opt_i(q_{j+1}) \leq y$

↳ a contradiction to (2)

$\therefore$ We deduce: $\boxed{opt_i(q_j) < x} \longrightarrow (4)$

By construction: we pick $s_j$ s.t. $opt_i(q_j) = rank(q_j, s_j)$

and consider the transition $q_j \xrightarrow{a_j} q'_{j+1}$ given

by $\tau(q_j, a_j, s_j)$

In proof of claim 1, we showed:

$$rank(q_j, s_j) \geqslant rank(q'_{j+1}, s_{j+1}) \quad \forall \text{ successors } s_{j+1} \text{ of } s_j$$

$$\therefore \boxed{opt_i(q_j) \geqslant rank(q'_{j+1}, r_{j+1})} \longrightarrow 5$$

where $r_{j+1}$ is the particular move in the memory token.

From 4 and 5

$$x > rank(q'_{j+1}, r_{j+1})$$

$$\therefore rank(q_j, r_j) \geqslant x > rank(q'_{j+1}, r_{j+1})$$

ie, $rank(q_j, r_j) > rank(q'_{j+1}, r_{j+1})$

**Case 2:** A Case 2 move looks like this in $\mathcal{G}_1(\mathcal{A}_i)$

$$
\begin{tikzcd}
\boxed{q_j, r_j} \arrow[d, "a_j", "2"'] \\
(q_j, a_j, r_j) \arrow[d, "1"] \\
\boxed{q_{j+1}, r_j, a_j} \arrow[d, "1/2"'] \\
\boxed{q_{j+1}, r_{j+1}}
\end{tikzcd}
$$

Priorities of intermediate transitions are indicated in blue.

$$\therefore \quad \text{rank}(q_j, r_j) \geq \text{rank}(q_{j+1}, r_{j+1})$$

**Case 1:** case 1 involves a move with priority 0.

Here there is no condition on the ranks.

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**Conclusion:** If Case 3 moves appear infinitely often, then Case 1 moves need to happen infinitely often too in order to reset the "ranks" to a higher value.

→ If Case 1 moves happen inf. often, Eve wins the play

→ Otherwise, after a point, only Case 2 moves happen starting from $(q_j, p_j, r_j)$.

→ Since Eve wins $G1(r_j, p_j)$, the run generated in memory token is accepting.

→ since Eve wins $G1(q_j, r_j)$, the run generated in Eve's token is accepting as well.

∴ Eve wins the play.

Part 2: Invariants are preserved after Step 3.

Recall:

Step 3: Obtain $\mathcal{A}_{i+1}$ from $\mathcal{A}_i'$ by changing priorities of transitions

$$q \xrightarrow{a:1} q'$$

s.t. $opt_i(q) > opt_i(q')$ to 0.

- Some of the priority 1 edges are changed to priority 0.
- To prove I3, it is sufficient to show that:

  Eve wins $G1((\mathcal{A}_{i+1}, q), (\mathcal{A}_{i+1}, p))$ for all $p, q$ that are WCR.

We already know that Eve wins $G1((\mathcal{A}_i', q), (\mathcal{A}_i', p))$

$\therefore$ It is sufficient to show the following:

Claim: a run is accepting in $\mathcal{A}_i'$ iff it is accepting in $\mathcal{A}_{i+1}$

→ Then Eve can play the same strategy as in $\mathcal{A}_i'$ in $\mathcal{A}_{i+1}$ as well.

Proof of claim: If it is accepting in $\mathcal{A}_i'$, then clearly it is accepting in $\mathcal{A}_{i+1}$.

Suppose a run is accepting in $\mathcal{A}_{i+1}$.

→ We will show that edges whose priorities were modified from 1 to 0 can occur only finitely often.

Let $\rho : q_0 \xrightarrow{a_0 : c_0} q_1 \xrightarrow{a_1 : c_1} q_2 \xrightarrow{a_2 : c_2} \cdots$

be a run of $\mathcal{A}_{i+1}$

If it is accepting, there are infinitely many $i$ s.t. $c_i = 0$

→ Due to step 2, for every transition

$$q_j \xrightarrow{a_j : 1} q_{j+1} \quad \text{with priority } 1$$

$$opt_i(q_j) \geq opt_i(q_{j+1})$$

Otherwise it would have been removed in step 2.

→ If $q_j \xrightarrow{a_j : 0} q_{j+1}$ is a transition which was modified in step 3, then:

$$opt_i(q_j) > opt_i(q_{j+1})$$

∴ These transitions cannot occur infinitely often unless

there are 0-priority transitions of $\mathcal{A}_i'$ occurring infinitely often,

as $opt_i$ is bounded.

## Part 3: Stabilization

Let $\mathcal{A}_N$ be the automaton with which the rank-reduction procedure terminates.

That is: Step 2 / step 3 cannot be performed anymore on $\mathcal{A}_N$.

**Claim:** $opt_N(q) = 0$ for all states $q$.

**Proof.**

Suppose $\exists q$ s.t. $opt_N(q) > 0$

$\therefore$ for all states $p \in WCR(\mathcal{A}_N, q)$

$rank_N(q,p) > 0$ in the game $\mathcal{G}_1(\mathcal{A}_N)$.

Since I1 and I2 are preserved during the rank-reduction procedure, we know that Eve wins $G1(q,p)$ in $\mathcal{A}_N$.

→ Since $rank_N(q,p) > 0$, there will be some priority 1 edge in the play of $\mathcal{G}_1(\mathcal{A}_N)$ before a priority 0 edge.

→ Recall priority 1 edges in $\mathcal{G}_1(\mathcal{A}_N)$ look like this:

$$(q_j, a_j, p_j)$$

$$\downarrow$$

$$(q_j', p_j, a_j) \dashrightarrow (q_j', p_j')$$

$q_j \xrightarrow{a_j} q_j'$ is a transition of $\mathcal{A}_N$.

→ By monotonicity of ranks,

$$rank_N(q_j, p_j) > rank_N(q_j', p_j')$$

Since we started with $(q,p)$ s.t. $\text{rank}_N(q,p) = \text{opt}_N(q)$

along the play, $\text{opt}_N$ should decrease along a priority 1

transition

→ But such transitions would be made priority 0 by

step 3.

→ Since $\mathcal{A}_N$ was stable, this is not possible.

Hence $\text{rank}_N(q,p) = 0 \Rightarrow \text{opt}_N(q) = 0$.