HISTORY DETERMINISM

Lecture 12: Characterizing History-Determinism in Büchi automata - Part II

Reference: Section 5.2 of Keya Prakash's Ph.D. thesis

In the previous lecture, we showed that $\mathcal{A}$ is a Büchi automaton s.t.

- 1. Eve wins the 1-token game from everywhere
- 2. $\mathcal{A}$ has reach-covering

Then $\mathcal{A}$ is HD.

In this lecture, we will consider an $\mathcal{A}$ where only (1) is true and modify it in such a way so that (2) holds.

Part 1: The game $\mathcal{G}_1(\mathcal{A})$: arena for 1-token games from everywhere in $\mathcal{A}$.

Part 2: Ranks in parity games

Part 3: Ranks and reach-covering

Part 4: The rank-reduction procedure

Part 1: The game $\mathcal{G}_1(\mathcal{A})$

This is a game which represents an arena for playing the 1-token games from all weakly-coreachable states in $\mathcal{A}$.

Nodes: partitioned as V_1, V_2, V_3

V_1 : $\boxed{q, p}$ $\quad q, p$ are WCR in $\mathcal{A}$

Adam's vertices

V_2 : (q, a, p) $\quad (q,p) \in V_1, \quad a \in \Sigma$

Eve's vertices

V_3 : $\boxed{q', p, a}$ $\quad (q, a, p) \in V_2$ and $q \xrightarrow{a:c} q' \in \Delta$

transition relation of $\mathcal{A}$.

Adam's vertices.

$V_\exists = V_2$ $\quad$ $V_\forall = V_1 \cup V_3$

Eve $\quad$ Adam

Edges:

E_1: $(q, p) \longrightarrow (q, a, p)$ (Adam chooses a letter)

E_2: $(q, a, p) \longrightarrow (q', p, a)$ (Eve chooses a transition on her token)

s.t. $q \xrightarrow{a:c_e} q' \in \Delta$

E_3: $(q', p, a) \longrightarrow (q', p')$ (Adam chooses a transition on his token)

s.t. $p \xrightarrow{a:c_a} p' \in \Delta$

Priorities:

- All edges in E_1 have priority 2.
- E_2: $(q, a, p) \longrightarrow (q', p, a)$ has priority 0 if $c_e = 0$
  else 2 if $c_e = 1$
- E_3: $(q', p, a) \longrightarrow (q', p')$ has priority 1 if $c_a = 0$
  2 if $c_a = 1$

Notice that a play starting from (q, p) is winning for Eve iff

(0 appears infinitely often) or (1 appears only finitely often)

iff.

Eve wins $G1(q, p)$.

Remarks:

- Winning region of Eve in the $[0,2]$-parity game $\mathcal{G}_1(\mathcal{A})$ is the set of all vertices.

  $\hookrightarrow$ since by hypothesis, Eve wins $G1(q,p)$ $\forall q,p$ that are WCR

- There is a uniform positional winning strategy τ for Eve in her winning region.

  τ maps: $(q, a, p) \longmapsto$ a transition $\delta: q \xrightarrow{a} q'$

  So for all $u \in V_\exists$, $\tau(u)$ is a transition δ.

- Consider a node $(q, a, p) \in V_\exists$

  if there is an outgoing edge $q \xrightarrow{a:\ 0} q'$ with priority 0 from q, then Eve can safely take it, since Eve wins from everywhere in $\mathcal{G}_1(\mathcal{A})$.

$\therefore$ WLOG, we assume that $\tau(u)$ is a 0-priority edge whenever there is an outgoing edge of priority 0 from q.

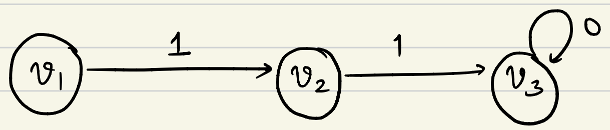
Part 2: Ranks in $[0,2]$-parity games.

Consider a $[0,2]$-parity game.

For a vertex v,

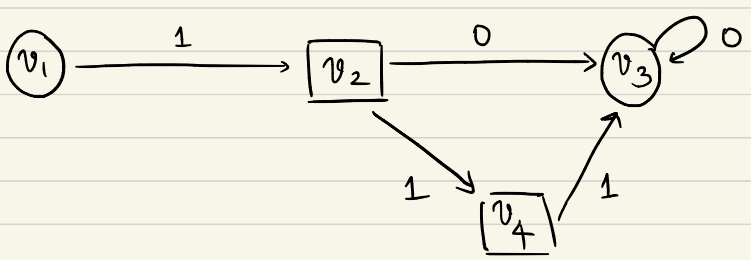
- $\mathrm{rank}(v)$ is a natural number or $+\infty$.
- $\mathrm{rank}(v)$ is the largest number s.t. Adam has a strategy to ensure at least $\mathrm{rank}(v)$ visits to a priority 1 edge before seeing a priority 0-edge.

Examples:

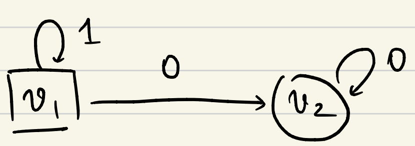


$\mathrm{rank}(v_1) = 2$ $\quad$ $\mathrm{rank}(v_2) = 1$ $\quad$ $\mathrm{rank}(v_3) = 0$.

$\square$: Adam, $\bigcirc$: Eve.



$\mathrm{rank}(v_1) = 3$, $\quad$ $\mathrm{rank}(v_2) = 2$, $\quad$ $\mathrm{rank}(v_4) = 1$



$\mathrm{rank}(v_1) = +\infty$

$\mathrm{rank}(v_2) = 0$

Properties about ranks :

Since Eve wins from every vertex in $\mathcal{G}_1(A)$:

- 1. $\mathrm{rank}(v)$ is bounded by a constant K

- 2. Monotonicity of ranks:

Let τ be an optimal strategy for Eve in G.
Then for every vertex u in G, the following holds.

- i) if $u \in V_\exists$ and $\tau(u) := u \xrightarrow{c} v$ then:

either $c = 0$ or $\mathrm{rank}(u) \geq \mathrm{rank}(v)$

inequality is strict if $c = 1$

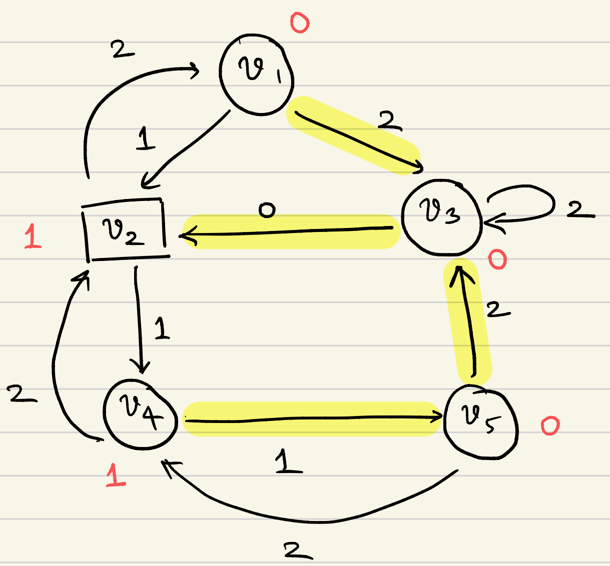
- i) if $u \in V_\forall$ then for all $u \xrightarrow{c} v$,

either $c = 0$ or $\mathrm{rank}(u) \geq \mathrm{rank}(v)$

inequality is strict if $c = 1$.

- 3. Playing optimal strategy τ ensures that in every play starting from v, at most $\mathrm{rank}(v)$ many edges of priority 1 are visited before visiting a priority 0 edge.

Example:



- Ranks are written in red next to the vertex.
- optimal strategy for Eve is shown in yellow.
- Notice the monotonicity property of ranks.

Part 3: Ranks and reach-covering

Recall that $\mathcal{A}$ has reach covering if $\forall q \ \exists p$ s.t.

Eve wins $G1^{reach}(q; p)$

i.e. Eve moves the token along a 0-edge before Adam moves along a 0-edge

Define:

$$\boxed{opt(q) = \min_{p \in WCK(\mathcal{A}, q)} rank(q, p)}$$

$\to$ in the game $\mathcal{G}_1(\mathcal{A})$.

Observe: $\mathcal{A}$ has reach-covering iff $opt(q) = 0 \ \forall q$.

Part 4: Rank-reduction procedure.

- Start with a Büchi automaton $\mathcal{A}$ on which Eve wins 1-token game from everywhere.
- We iteratively modify $\mathcal{A}$ to $\mathcal{A}_N$, so that $opt(q) = 0 \ \forall q$ in $\mathcal{A}_N$.

Set $\mathcal{A}_0 = \mathcal{A}$. For each $i \geq 0$, we perform the following three steps on $\mathcal{A}_i$ until $\mathcal{A}_{i+1} = \mathcal{A}_i$

Step 1: For each state q in $\mathcal{A}_i$, compute the optimal rank of q in $\mathcal{G}_1(\mathcal{A}_i)$, which we denote $opt_i(q)$.

Step 2: Obtain $\mathcal{A}_i'$ from $\mathcal{A}_i$ by removing all transitions

$$q \xrightarrow{a:1} q'$$

with $opt_i(q) < opt_i(q')$

Step 3: Obtain $\mathcal{A}_{i+1}$ from $\mathcal{A}_i'$ by changing priorities of transitions

$$q \xrightarrow{a:1} q'$$

s.t. $opt_i(q) > opt_i(q')$ to 0.

In the next lecture, we will argue the "correctness" of the rank-reduction procedure.