HISTORY DETERMINISM

Lecture 11: Characterizing History-Determinism in Büchi automata

Reference: Section 5.1 of Keya Prakash's Ph.D. thesis

Büchi automata are [0,1]-parity automata with min-parity-even condition.

Our objective is to show the following theorem.

Theorem: Eve wins the Joker game in a Büchi automaton $\mathcal{A}$

iff

$\mathcal{A}$ is HD.

As in the coBüchi case, what remains to be shown is the following:

Theorem: Let $\mathcal{A}$ be a Büchi automaton where Eve wins the 1-token game from everywhere. Then $\mathcal{A}$ is HD.

Here are the steps to the proof.

Step 1: Obtaining the reachability automaton $\mathcal{A}_{reach}$ from $\mathcal{A}$, and the idea of "reach-coverage"

Step 2: Suppose Eve wins 1-token game everywhere on $\mathcal{A}$ + $\mathcal{A}$ has reach-coverage, then $\mathcal{A}$ is HD.

Step 3: From $\mathcal{A}$, getting a simulation-equivalent automaton which has reach-coverage

We will discuss Steps 1 & 2 in this lecture.

Step 1: From $\mathcal{A}$ to $\mathcal{A}_{reach}$.

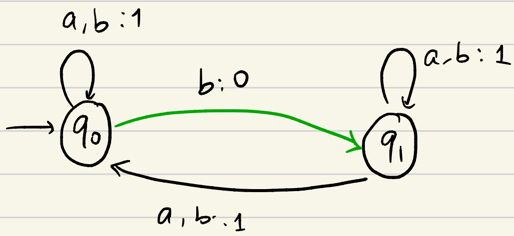
$\mathcal{A}_{reach}$:
- same states as that of $\mathcal{A}$ + an "accepting" sink state q_T
- all 1-transitions of $\mathcal{A}$ remain in $\mathcal{A}_{reach}$
- for every $q \xrightarrow{a:0} q'$ in $\mathcal{A}$ we have:

$$q \xrightarrow{a:0} q_T \text{ in } \mathcal{A}_{reach}.$$

- $q_T \xrightarrow{a:0} q_T \quad \forall a.$

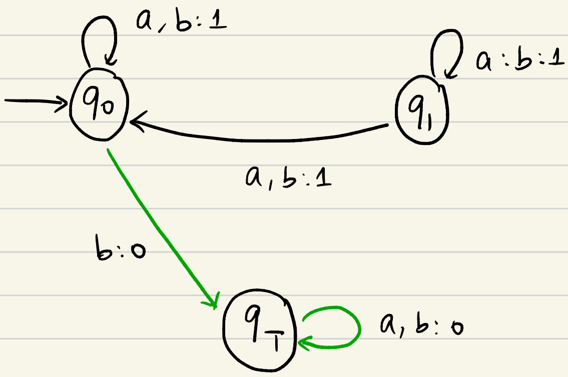
Examples:

$\mathcal{A}^1$

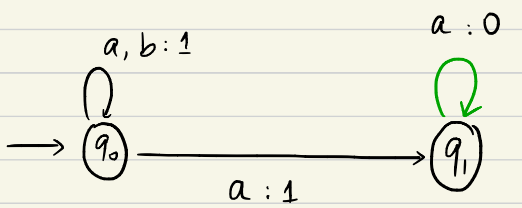


$\mathcal{L}(\mathcal{A}^1)$: set of ω-words where b occurs infinitely often.

$\mathcal{A}^1_{reach}$:

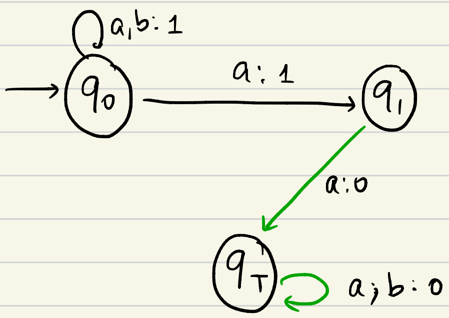


$\mathcal{A}^2$:



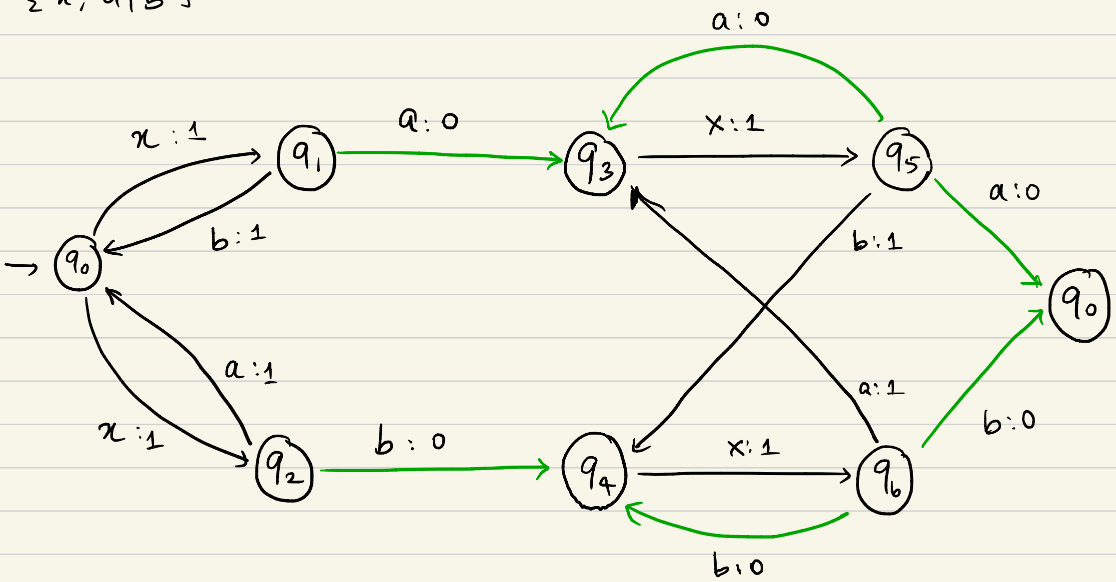
$\mathcal{L}(\mathcal{A}^2)$: b occurs only finitely often.

$\mathcal{A}^2_{reach}$:



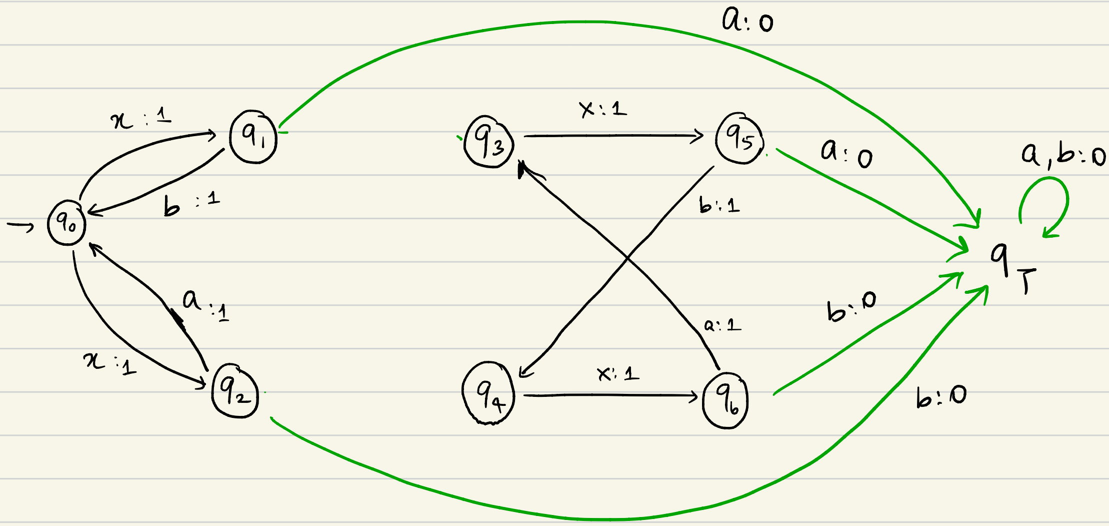
$\Sigma = \{x, a, b\}$

$\mathcal{A}^3$:



$\mathcal{L}(\mathcal{A}^3)$: $\left((xa + xb)^* (xaxa + xbxb)\right)^\omega$

$\mathcal{A}^3_{reach}$:

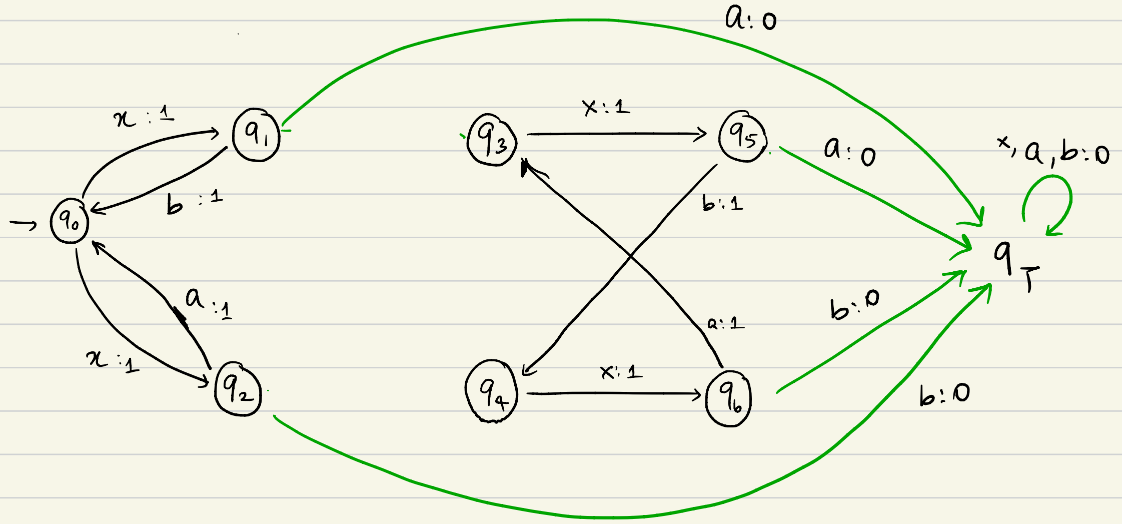


$\mathcal{A}_{reach}$ is a reachability automaton: an ω-word is accepted if it reaches the accepting sink state

It is easy to see the following observation.

Lemma: $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}_{reach})$

Example: What is $\mathcal{L}(\mathcal{A}^3_{reach})$?



$$(xa + xb)^* (xa + xb) . \Sigma^\omega$$

Reach-covering

We say that a Büchi automaton $\mathcal{A}$ has reach-covering if $\forall$ states q, $\exists\, p \in WCR(\mathcal{A}, q)$ s.t. Eve wins $G1\,(q;\, p)$ in $\mathcal{A}_{reach}$

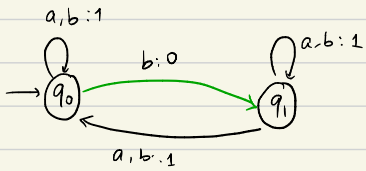
Notice that it says $G1\,(q;\, p)$ in $\mathcal{A}_{reach}$.

↳ starting in $\mathcal{A}_{reach}$ from $(q;\, p)$, if Adam has a way to reach the sink state, then Eve can force her token to the sink state too.

Remark: For the definition of safe-coverage, it was $G1\,(\underline{\underline{p}};\, q)$ — whereas here it is $G1\,(q;\, p)$

Examples:

$\mathcal{A}^1$



$\mathcal{A}'$ has reach covering

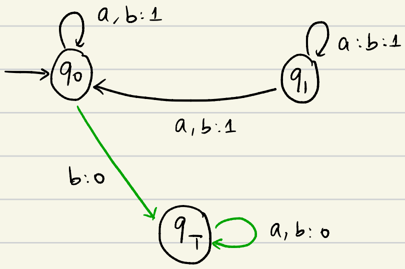
Eve wins: $G1^{reach}\,(q_0, q_0)$

$G1^{reach}\,(q_1, q_1)$

$G1^{reach}\,(q_0, q_1)$

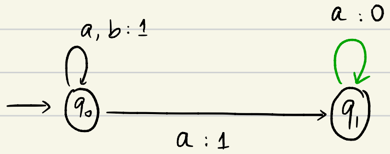
$G1^{reach}\,(q_1, q_0)$

$\mathcal{A}^1_{reach}$:



→ $\mathcal{A}^2$ does not have reach-covering.

$\mathcal{A}^2$:

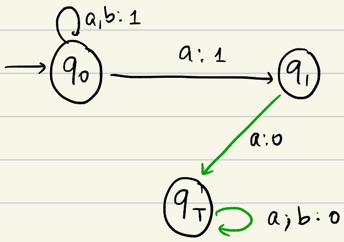


Adam wins

$G_1^{reach}(q_0 ; q_0)$

and $G_1^{reach}(q_0 ; q_1)$

$\mathcal{A}^2_{reach}$:



Step 2: Reach covering

Theorem: Let $\mathcal{A}$ be a Büchi automaton with n states, s.t.

- Eve wins the 1-token game from everywhere, and
- $\mathcal{A}$ has reach-covering.

I. Then $\mathcal{A}$ is H.D.

II. Moreover, a language-equivalent deterministic automaton with at most n^2 states can be computed in PTIME.

Proof:

Part 1: From the reach-covering property, we can deduce that:

$$\forall q \ \exists p \quad \text{s.t. Eve wins } G1^{reach}(q;p) \ \underbrace{\text{and } G1^{reach}(p;p)}_{}$$

Proof of this part is similar to the coBüchi case.

Reach-deterministic states:

↳ States 'p' s.t. Eve wins $G1^{reach}(p;p)$

Therefore we can say that

$\forall q \ \exists p$ s.t. p is reach-deterministic &

Eve wins $G1^{reach}(q;p)$

Part 2: Uniform positional winning strategy.

Consider the following arena:

Nodes: □ belong to Adam; ○ belong to Eve

$\boxed{q;\ p}$ (q, p, a) $\boxed{q', p, a}$

for all pairs q, p that are HICR in $\mathcal{A}$

$a \in \Sigma$

Edges:

$\boxed{q;\ p} \xrightarrow{a} (q, p, a) \longrightarrow \boxed{q'; p, a} \longrightarrow \boxed{q', p'}$

Adam chooses letter

s.t. $q \xrightarrow{a} q'$ in $\mathcal{A}_{reach}$

Eve chooses transition

s.t. $p \xrightarrow{a} p'$ in $\mathcal{A}_{reach}$

Adam chooses transition

The above sequence represents one round of the 1-token game in $\mathcal{A}_{reach}$.

Winning condition: same as 1-token winning condition of $\mathcal{A}_{reach}$

→ can be represented as a 2-dimensional parity condition.

∴ Eve has a uniform winning strategy in its winning region.

Winning region: contains the following nodes:

- all $\boxed{p, p}$ s.t. p is reach-deterministic
- $\forall q\ \exists p$ in reach-deterministic s.t. $\boxed{q, p}$ is in winning reg.

Let τ be uniform positional winning strategy for Eve in this game.

Part 3: The automaton $\mathcal{D}_{reach}$.

Similar to the coBüchi case, we have a deterministic reachability automaton $\mathcal{D}_{reach}$

- whose states are all reach-deterministic states of $\mathcal{A}_{reach}$ (including the sink)
- $\boxed{\forall \text{ reach-deterministic states } `p', \quad \mathcal{L}(\mathcal{D}_{reach}, p) = \mathcal{L}(\mathcal{A}_{reach}, p)}$

Part 4: The required deterministic automaton $\mathcal{D}$.

We now describe a deterministic Büchi automaton $\mathcal{D}$ that is language equivalent to $\mathcal{A}$.

States of $\mathcal{D}$: all pairs (q, r) s.t.

- Eve wins $G1^{reach}(q, r)$ and

- i) either r is reach-deterministic and $r \in WCR(\mathcal{A}, q)$
- ii) or r is the accepting sink state of $\mathcal{A}_{reach}$

Initial state: (q_0, r_0) s.t. r_0 is a reach-deterministic state in $WCR(\mathcal{A}, q_0)$

If there are multiple possibilities for r_0 we choose & fix one of them arbitrarily

<u>Transitions:</u> Consider a state (q, r) and $a \in \Sigma$.

- Let $\delta = q \xrightarrow{a:c} q'$ be the transition given by strategy τ.

  ie, $\tau(q, r, a) = \delta$

i) if $c = 0$, ie., δ is an accepting transition in $\mathcal{A}$, we have

$$(q, r) \xrightarrow{a:0} (q', r')$$

where r' is a reach-deterministic state in $WCR(\mathcal{A}, q')$.

We fix this choice arbitrarily if there are multiple possibilities.

ii) Otherwise we add transition

$$(q, r) \xrightarrow{a:1} (q', r')$$

where $r \xrightarrow{a} r'$ is the unique transition in $\mathcal{D}reach$.

$\mathcal{L}(\mathcal{D}) \subseteq \mathcal{L}(\mathcal{A})$: Since the colours are inherited from the first component, the projection of an accepting run of $\mathcal{D}$ to its first component gives an accepting run of $\mathcal{A}$.

This proves this direction.

$\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{D})$:

Let $w \in \mathcal{L}(\mathcal{A})$

Let $\rho_{\mathcal{D}}$ be the unique run of $\mathcal{D}$ on w.

We show that if $\rho_{\mathcal{D}}$ is rejecting, we will hit a contradiction.

Suppose $\rho_{\mathcal{D}}$ is rejecting.

$$\rho_{\mathcal{D}} := (q_0, r_0) \xrightarrow{u} (q, r) \xrightarrow{w'} \cdots$$

w can be written as uw' s.t. the run on w' visits no 0-transition at all.

In particular, by definition of the transition relation, the projection of the run: $(q, r) \xrightarrow{w'} \cdots$ to the second component gives the unique run of $\mathcal{D}_{reach}$ on w' starting from r.

If $w' \in \mathcal{L}(\mathcal{D}_{reach}, r) = \mathcal{L}(\mathcal{A}_{reach}, r)$, the strategy τ would ensure that the first component is also accepting.

However it is not.

Hence we deduce $w' \notin \mathcal{L}(\mathcal{A}_{reach}, r)$

Since $\mathcal{L}(\mathcal{A}, r) \subseteq \mathcal{L}(\mathcal{A}_{reach}, r)$

We conclude:

$$w' \notin \mathcal{L}(\mathcal{A}, r) \longrightarrow (1)$$

Now, since $w \in \mathcal{L}(\mathcal{A})$, there is an accepting run, say:

$$\rho_A : \quad q_0 \xrightarrow{u} q_u \xrightarrow{w'}$$

Here, $w' \in \mathcal{L}(\mathcal{A}, q_u)$

Notice that $q \in \mathrm{WCR}(\mathcal{A}, q_u)$ and hence

$$r \in \mathrm{WCR}(\mathcal{A}, q_u).$$

Since Eve wins 1-token game everywhere in $\mathcal{A}$,

Eve wins $G1^{\mathcal{A}}(r; q_u)$

Hence $w' \in \mathcal{L}(\mathcal{A}, r) \longrightarrow$ 2

From 1 and 2 we get a contradiction.

This proves that ρ_D should be accepting.