1. Consider the following problem.

$$\begin{array}{lrcrcrcrcrcl}
\text{Maximize} & 10x_1 & + & 24x_2 & + & 20x_3 & + & 20x_4 & + & 25x_5 & & \\
\text{Subject to} & x_1 & + & x_2 & + & 2x_3 & + & 3x_4 & + & 5x_5 & \leq 19 & (C1) \\
 & 2x_1 & + & 4x_2 & + & 3x_3 & + & 2x_4 & + & x_5 & \leq 57 & (C2) \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq 0 &
\end{array}$$

   (a) Write its dual with two variables w_1, w_2 (corresponding to the constraints (C1) and (C2)) and verify that $(w_1, w_2) = (4, 5)$ is a feasible solution.

   (b) Use complementary slackness to show that $(w_1, w_2) = (4, 5)$ gives the optimal solution to the dual.

2. Consider the following problem:

$$\begin{array}{lrcrcr}
\text{Maximize} & 9x_1 & + & 8x_2 & & \\
\text{subject to} & x_1 & - & 2x_2 & \leq & -1 \\
 & 4x_1 & + & 3x_2 & \leq & 4 \\
 & -x_1 & + & 2x_2 & \leq & 3 \\
 & 2x_1 & - & x_2 & \leq & -4
\end{array}$$

   Verify, using complementary slackness, whether $x_1 = -3, x_2 = -1$ is optimal. Verify using complementary slackness whether $x_1 = -\frac{5}{3}, x_2 = \frac{2}{3}$ is optimal.

3. Suppose we allow negative edge weights in the shortest paths problem discussed in class. Prove that the following conditions are equivalent:

   (a) There is a shortest path from s to t.

   (b) The dual LP is feasible.

   (c) There is no cycle with negative total cost.

4. Here is the set cover problem:

   Input. A universe D consisting of finite number of elements, and a family $S_1, S_2, \ldots, S_m$ of sets, with each $S_i \subseteq D$. Assume that $\bigcup_{i \in \{1,\ldots,m\}} S_i = D$.

   Goal. Find minimum size subset $W \subseteq \{1, \ldots, m\}$ such that $\bigcup_{i \in W} S_i = D$

   (a) Write an ILP for the set cover problem.

   (b) Give the relaxed LP and its dual.

   (c) Assume that each element of D occurs in at most f sets among $S_1, \ldots, S_m$. Design a primal-dual algorithm which gives an f-approximation: that is, the answer given by the algorithm is at most f times the optimal solution.

   Describe the algorithm and its analysis.