Linear Programming & Combinatorial Optimization

Lecture 9

PROOF OF DUALITY VIA SIMPLEX

maximize $c^T x$

subject to $Ax \leq b$

$x \geq 0$

PRIMAL (P)

DUAL (D)

minimize $b^T y$

subject to $A^T y \geq c$

$y \geq 0$

WEAK DUALITY:

For every feasible solution $\bar{x}$ of (P),
for every feasible solution $\bar{y}$ of (D):

$$c^T \bar{x} \leq b^T \bar{y}$$

STRONG DUALITY:

Exactly one of the foll. occurs:

- 1. Both (P) and (D) infeasible
- 2. (P) unbounded, (D) infeasible
- 3. (P) infeasible, (D) unbounded
- 4. Optimum (P) = x^*, Optimum (D) = y^*

$$\boxed{c^T x^* = b^T y^*}$$

(P) \ (D)	infeasible	unbounded	∃ optimum
infeasible	✓	✓	✗ (S)
unbounded	✓	✗ (W)	✗ (W)
∃ optimum	✗ (S)	✗ (W)	✓

Exercise:

Write the dual of the following LP:

maximize $5 + 3x_1 + 4x_2 - 2x_3$

subject to
$$\begin{aligned} x_1 - 7x_2 + 5x_3 &\leq 10 \\ 2x_1 + 3x_2 - x_3 &\leq 15 \\ x_2 + 7x_3 &\leq 20 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Dual:

minimize $5 + 10y_1 + 15y_2 + 20y_3$

subj. to:
$$\begin{aligned} y_1 + 2y_2 &\geq 3 \\ -7y_1 + 3y_2 + y_3 &\geq 4 \\ 5y_1 - y_2 + 7y_3 &\geq -2 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

GOAL: Proof of duality theorem, using simplex

REFERENCE: Section 6.3 of text:

Understanding and Using Linear Programming
— Matoušek & Gärtner

We will prove the following:

> When primal has an optimum,
> - the dual is feasible and
> - optimum of dual coincides with optimum of primal

STEP 1: Consider primal to be in equational form

Primal		Dual	
max	$c^T x$	minimize	$b^T y$
subject to	$Ax = b$	subject to	$A^T y \geq c$
	$x \geq 0$		y unrestricted

Primal   Dual

Exercise:

$$\begin{tikzcd}
\text{Primal } 1 \arrow[r, mapsto] \arrow[d, "\text{equational form}"'] & \text{Dual } 1 \\
\text{Primal } 2 \arrow[r, mapsto] & \text{Dual } 2 \arrow[u]
\end{tikzcd}$$

Dual 2 feasible
<=>
Dual 1 feasible

Optimal cost (dual 1)
= optimal cost (dual 2)

STEP 2: Elementary operations preserve the dual optimum

max $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

$x_1, x_2, \ldots, x_n \geq 0$

min $b_1y_1 + b_2y_2 + \cdots + b_my_m$

$$a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1$$
$$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq c_2$$
$$\vdots$$
$$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n$$

(D1)

$y_1, y_2, \ldots, y_m$ unrestricted

Multiplying a primal equation by a scalar.

$$\vdots$$
$$\alpha a_{i1}x_1 + \alpha a_{i2}x_2 + \cdots + \alpha a_{in}x_n = \alpha b_i$$
$$\vdots$$

Set of solutions does not change

min $\cdots + \alpha b_i y_i + \cdots$

$$\cdots + \alpha a_{i1}y_i + \cdots \geq c_1$$
$$+ \alpha a_{i2}y_i + \cdots \geq c_2$$
$$\vdots$$
$$+ \alpha a_{in}y_i + \cdots \geq c_n$$

(D2)

$y_1, y_2, \ldots, y_m$ unrestricted.

Exercise: (D1) feasible $\iff$ (D2) feasible; Optimal costs are the same for both (D1) & (D2)

$(y_1', \ldots, y_m')$ is a soln. of (D1) $\iff$ $(y_1', \ldots \frac{y_i'}{\alpha}, \ldots y_m')$ soln. of (D2)

Replacing a primal equation i by sum of equations i and j

max $c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1$

$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = b_2$

$\vdots$

$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m$

$x_1, x_2, \dots, x_n \geq 0$

(D1)

min $b_1 y_1 + b_2 y_2 + \dots + b_m y_m$

$a_{11} y_1 + a_{21} y_2 + \dots + a_{m1} y_m \geq c_1$

$a_{12} y_1 + a_{22} y_2 + \dots + a_{m2} y_m \geq c_2$

$\vdots$

$a_{1n} y_1 + a_{2n} y_2 + \dots + a_{mn} y_m \geq c_n$

$y_1, y_2, \dots y_m$ unrestricted

$\vdots$

$(a_{i1} + a_{j1}) x_1 + \dots + (a_{in} + a_{jn}) x_n = b_i + b_j$

(D2)

min $\dots + (b_i + b_j) y_i + \dots$

$+ (a_{i1} + a_{j1}) y_i + \dots$

$\vdots$

$+ (a_{in} + a_{jn}) y_i + \dots$

Exercise: (D1) feasible $\iff$ (D2) feasible, optimal costs are the same for both (D1) & (D2)

$(y_1', y_2', \dots, y_m')$ is a soln. to (D1) $\rightarrow$ $(y_1', \dots, \underbrace{y_j' - y_i'}_{j^{th}\text{ coord.}}, \dots y_i', \dots)$

$\leftarrow$

$$\min \quad b_1 y_1 + b_2 y_2 + \cdots + b_m y_m$$

$$a_{11} y_1 + a_{21} y_2 + \cdots + a_{m1} y_m \geq c_1$$

$$a_{12} y_1 + a_{22} y_2 + \cdots + a_{m2} y_m \geq c_2$$

(D1)

$$\vdots$$

$$a_{1n} y_1 + a_{2n} y_2 + \cdots + a_{mn} y_m \geq c_n$$

$y_1, y_2, \ldots y_m$ unrestricted

$$\min \quad \cdots + (b_i + b_j) y_i + \cdots$$

$$+ (a_{i1} + a_{j1}) y_i + \cdots$$

(D2)

$$\vdots$$

$$+ (a_{in} + a_{jn}) y_i + \cdots$$

c_j

$$a_{j1} \underbrace{(y_i + y_j)} + a_{i1} y_i$$

$$y_i + y_j = y_j' \quad \Rightarrow \quad y_j = y_j' - y_i'$$

$$y_i = y_i' \qquad\qquad y_i = y_i'$$

STEP 3. Observe that simplex tableaus are obtained through a sequence of elementary operations starting from the original system of equations

$$Ax = b \qquad x \geq 0$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\Big\downarrow \text{ elementary operations}$$

Basis $\longrightarrow$

$$\begin{array}{c|c} & \begin{matrix} \cdots & i_1 & \cdots & i_2 & \cdots & i_m \end{matrix} \\ \begin{matrix} \\ i_1 \\ i_2 \\ \vdots \\ \\ i_m \end{matrix} & \begin{bmatrix} \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \cdots & 0 & \cdots & 1 & \cdots & \vdots \\ \cdots & 0 & \cdots & 0 & \cdots & \vdots \\ \cdots & 0 & \cdots & \vdots & \cdots & \\ \cdots & 0 & \cdots & \cdot & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1' \\ b_2' \\ \vdots \\ b_m' \end{bmatrix} \end{array}$$

$$\Big\downarrow$$

$$x_B = p + Q\,x_N$$

$$z = z_0 + r^T x_N$$

non-basic

$$x_{i_1} = p_1 + \boxed{\cdots\cdots}$$
$$x_{i_2} = p_2 + \boxed{}$$
$$\vdots$$
$$x_{i_m} = p_m + \boxed{}$$

From Step 2 and Step 3: Dual optimum of original system is the same as dual optimum of a simplex tableau corresponding to original system

Step 4: What is the dual when primal is given by a simplex tableau?

$$x_B = p + Q\,x_N$$

$$B = \{ i_1, i_2, \dots, i_m \} \qquad N = \{ j_1, j_2, \dots, j_{n-m} \}$$

$$Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1,n-m} \\ \vdots & & & \vdots \\ q_{m1} & \cdots & \cdots & q_{m,n-m} \end{bmatrix} \begin{matrix} x_{j_1} \\ \vdots \\ x_{j_{n-m}} \end{matrix}$$

Equations:

$$x_{i_1} - q_{11}x_{j_1} - q_{12}x_{j_2} - \dots - q_{1(n-m)}x_{j_{n-m}} = p_1$$

$$x_{i_2} - q_{21}x_{j_1} - q_{22}x_{j_2} - \dots - q_{2(n-m)}x_{j_{n-m}} = p_2$$

$$\vdots$$

$$x_{i_m} - q_{m1}x_{j_1} - q_{m2}x_{j_2} - \dots - q_{m(n-m)}x_{j_{n-m}} = p_m$$

Cost: $c_1x_1 + c_2x_2 + \dots + c_nx_n = c_{i_1}x_{i_1} + c_{i_2}x_{i_2} + \dots + c_{i_m}x_{i_m} + c_{j_1}x_{j_1} + \dots + c_{j_{n-m}}x_{j_{n-m}}$

↳ can be rewritten using $j_1 \dots j_{n-m}$

$$\boxed{= \underbrace{(c_{i_1}p_1 + c_{i_2}p_2 + \dots + c_{i_m}p_m)}_{z_0} + \underbrace{\underbrace{(c_{j_1} + c_{i_1}q_{11} + c_{i_2}q_{21} + \dots + c_{i_m}q_{m1})}_{r_1}x_{j_1} + \dots + (c_{j_{n-m}} + c_{i_1}q_{1(n-m)} + \dots + c_{i_m}q_{m(n-m)})x_{j_{n-m}}}_{r^T x_N}}$$

Dual constraints:

$$y_1 \geq 0,\ y_2 \geq 0, \dots,\ y_m \geq 0$$

$$-q_{11}y_1 - q_{21}y_2 + \dots - q_{m1}y_m \geq r_1$$

$$\vdots$$

$$-q_{1(n-m)}y_1 \dots - q_{m(n-m)}y_m \geq r_{n-m}$$

Equations:

$$x_{i_1} - q_{11} x_{j_1} - q_{12} x_{j_2} - \cdots - q_{1(n-m)} x_{j_{n-m}} = p_1$$
$$x_{i_2} - q_{21} x_{j_1} - q_{22} x_{j_2} - \cdots - q_{2(n-m)} x_{j_{n-m}} = p_2$$
$$\vdots$$
$$x_{i_m} - q_{m1} x_{j_1} - q_{m2} x_{j_2} - \cdots - q_{m(n-m)} x_{j_{n-m}} = p_m$$

Cost: $c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = c_{i_1} x_{i_1} + c_{i_2} x_{i_2} + \cdots + c_{i_m} x_{i_m} + c_{j_1} x_{j_1} + \cdots + c_{j_{n-m}} x_{j_{n-m}}$

↳ can be rewritten using $j_1 \cdots j_{n-m}$

$$\boxed{= \underbrace{(c_{i_1} p_1 + c_{i_2} p_2 + \cdots + c_{i_m} p_m)}_{z_0} + \underbrace{(c_{j_1} + c_{i_1} q_{11} + c_{i_2} q_{21} + \cdots + c_{i_m} q_{m1}) x_{j_1} + \cdots + (c_{j_{n-m}} + c_{i_1} q_{1(n-m)} + \cdots + c_{i_m} q_{m(n-m)}) x_{j_{n-m}}}_{r^T x_N}}$$

Dual constraints:

$$y_1, y_2, \ldots, y_m \geq 0$$
$$-q_{11} y_1 - q_{21} y_2 \cdots - q_{m1} y_m \geq r_1$$
$$\vdots$$
$$-q_{1(n-m)} y_1 - \cdots - q_{m(n-m)} y_m \geq r_{n-m}$$

STEP 5: Main observation: Assume primal has optimum.

In the optimal tableau, coefficients $r_1, r_2, \ldots r_{n-m}$ are negative!

Hence $y_1, y_2, \ldots, y_m = 0$ is a feasible solution to the dual constraints.

Dual cost: $z_0 + p_1 y_1 + p_2 y_2 + \cdots + p_m y_m$

$= z_0$ when $y_1, y_2, \ldots, y_m = 0$

By analyzing the final simplex tableau of the primal we get:

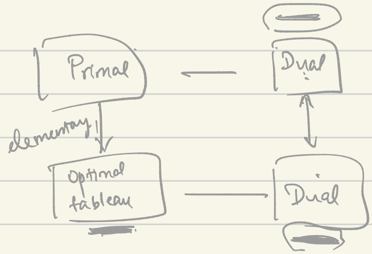
$y_1 = 0, \dots y_m = 0$ is a feasible soln. of dual with cost z_0.

Primal optimum is z_0.

- By weak duality, dual cost $\geq z_0$.
- We have a point in the dual with cost z_0.

$\Rightarrow$ optimum cost of dual $= z_0$.

What does this show?



When primal has an optimum:

- consider the final tableau
- Write the dual constraints for this system
- dual optimum equals the optimum of this system corresponding to final tableau

But, we have seen that elementary operations preserve the dual optimum.

We can also reconstruct the dual feasible solution giving the optimum for the original system.

- Proves duality theorem!

PROOF OF DUALITY USING FARKAS' LEMMA

REFERENCE: Section 6.4 of text:

Understanding and Using Linear Programming
— Matoušek & Gärtner

FARKAS' LEMMA:

Let A be a real matrix with m rows and n columns.
Let $b \in \mathbb{R}^m$ be a vector.

Then exactly one of the following two possibilities occur:

(F1) There exists a vector $x \in \mathbb{R}^n$ satisfying $Ax = b$ and $x \geq 0$.
(F2) There exists a vector $y \in \mathbb{R}^m$ satisfying $y^T A \geq 0$ and $y^T b < 0$.

$$\begin{bmatrix} y_1 & \cdots & y_m \end{bmatrix} \overset{A}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix}}$$

$$\begin{bmatrix} y_1 a_{11} + y_2 a_{21} + \cdots y_m a_{m1} \\ \vdots \\ y_1 a_{1n} + \cdots \quad y_m a_{mn} \end{bmatrix}$$

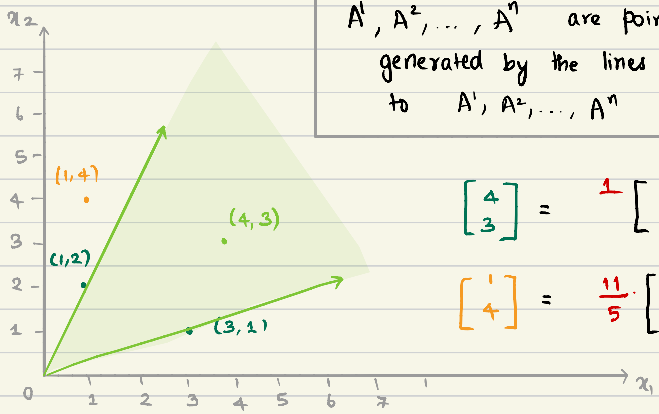
Non-negative solutions to $Ax = b$:

$$x_1 A^1 + x_2 A^2 + \cdots + x_n A^n = b$$

To understand: non-negative linear combinations of vectors

GEOMETRIC VIEW OF FARKAS' LEMMA:

Non-negative combinations of vectors $A^1, A^2, \dots, A^n$ are points in the cone generated by the lines joining origin to $A^1, A^2, \dots, A^n$



$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

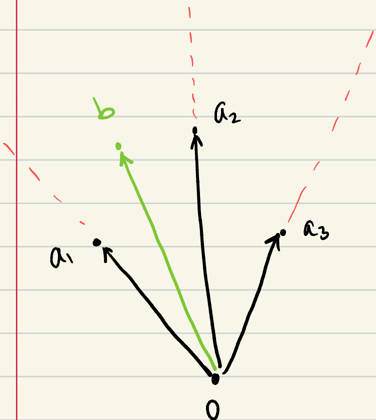
$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \frac{11}{5} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{2}{5} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

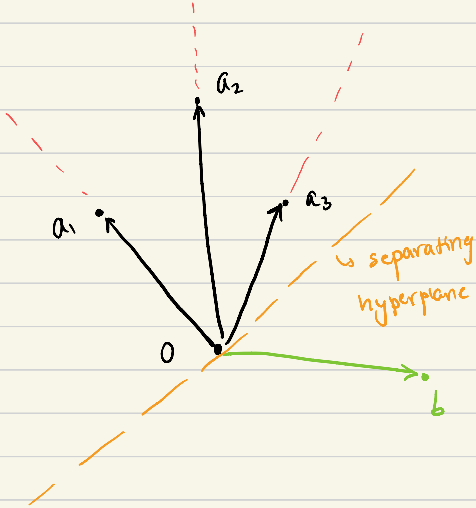
CONVEX CONE:

Given vectors $a_1, a_2, \dots, a_n \in \mathbb{R}^m$, the convex cone generated by $a_1, a_2, \dots, a_n$ is the set of all linear combinations of the a_i using non-negative coefficients:

$$\{ t_1 a_1 + t_2 a_2 + \dots + t_n a_n \mid t_1, \dots, t_n \geq 0 \}$$

$Ax = b$ has a non-negative soln? $a_1, \dots a_n$ are columns of A





b lies inside the cone

b lies outside the cone

$Ax = b$ has a non-negative soln.

Separating hyperplane:

There is a hyperplane passing through 0:

Variables: $x_1, \dots x_m$

$$y_1 x_1 + \dots + y_m x_m = 0$$

s.t. $y a_i \geq 0$

$y b < 0$

Summary:

- Proof of duality through simplex
- Proof of duality through Farkas' lemma:
    - Statement of Farkas' lemma
    - Geometric interpretation of the statement
    - Proof of duality from F.L.
    - Proof of Farkas' lemma

} next class.