LINEAR PROGRAMMING & COMBINATORIAL OPTIMIZATION

LECTURE 8

DUAL OF AN LP

GOAL: Given an LP, associate another LP to it (called its "dual") which has interesting relations to the original LP

REFERENCE: Section 6.1, 6.2 of text:

Understanding and Using Linear Programming
— Matoušek & Gärtner

Part 1: Intuition:

maximize $2x_1 + 3x_2$

subject to
$$4x_1 + 8x_2 \leq 12 \quad (1)$$
$$2x_1 + x_2 \leq 3 \quad (2)$$
$$3x_1 + 2x_2 \leq 4 \quad (3)$$
$$x_1, x_2 \geq 0$$

$$2x_1 + 3x_2 \leq 4x_1 + 8x_2 \leq 12$$

$\Rightarrow$ Cost ≤ 12

$$2x_1 + 3x_2 \leq 1/2\,(4x_1 + 8x_2) \leq 1/2 \cdot 12$$

$\Rightarrow$ cost ≤ 6

Adding (1) and (2): $6x_1 + 9x_2 \leq 15$

$$2x_1 + 3x_2 \leq 1/3\,(6x_1 + 9x_2) \leq 5$$

Cost ≤ 5

Combining inequalities by scalar multiplication and addition gives a bound on the cost.

$$\begin{array}{rl} \cancel{\times} \quad \times & 4x_1 + 8x_2 \leq 12 \\ & 2x_1 + x_2 \leq 3 \\ & 3x_1 + 2x_2 \leq 4 \\ & x_1 , x_2 \geq 0 \end{array}$$

Multiply with a non-negative real

$$\begin{array}{rrl} & y_1 \times & 4x_1 + 8x_2 \leq 12 \\ + & y_2 \times & 2x_1 + x_2 \leq 3 \\ + & y_3 \times & 3x_1 + 2x_2 \leq 4 \\ & & x_1 , x_2 \geq 0 \end{array}$$

$y_1, y_2, y_3 \geq 0$

$$(4y_1 + 2y_2 + 3y_3)\, x_1 + (8y_1 + y_2 + 2y_3)\, x_2 \leq 12y_1 + 3y_2 + 4y_3$$

$\geq$ $\geq$

maximize $2x_1 + 3x_2$

Suppose we have y_1, y_2, y_3 s.t.

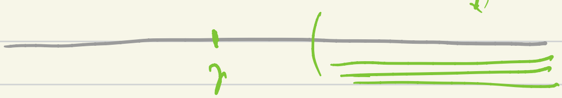
$$4y_1 + 2y_2 + 3y_3 \geq 2$$
$$8y_1 + y_2 + 2y_3 \geq 3$$
$$y_1, y_2, y_3 \geq 0$$

$\Rightarrow$ Cost (LP) $= 2x_1 + 3x_2 \leq 12y_1 + 3y_2 + 4y_3$

Suppose we have y_1, y_2, y_3 s.t.

$$4y_1 + 2y_2 + 3y_3 \geq 2$$
$$8y_1 + y_2 + 2y_3 \geq 3$$
$$y_1, y_2, y_3 \geq 0$$

$\Rightarrow$ $\text{Cost}(LP) = 2x_1 + 3x_2 \leq 12y_1 + 3y_2 + 4y_3$



Suppose $\text{Cost}(LP) = r$.

<u>Another LP over y_1, y_2, y_3:</u>

$$\left.\begin{aligned} &\text{minimize} && 12y_1 + 3y_2 + 4y_3 \\ &\text{s.t.} && 4y_1 + 2y_2 + 3y_3 \geq 2 \\ & && 8y_1 + y_2 + 2y_3 \geq 3 \\ & && y_1, y_2, y_3 \geq 0 \end{aligned}\right\} \text{Dual}$$

Intuitively optimum of above LP gives an upper bound on the optimum of original LP.

Exercise: Write the dual for the following LPs.

1) maximize: $12x_1 + 3x_2 + 5x_3$

$$\begin{aligned} 7x_1 + 4x_2 + 2x_3 &\leq 100 \\ 2x_1 \qquad\quad + 7x_3 &\leq 50 \\ x_1, \; x_2, \; x_3 &\geq 0 \end{aligned}$$

2) maximize $2x_1 + 3x_2$

subject to:

$$\begin{aligned} 4x_1 + 8x_2 &\leq 12 \\ 2x_1 + x_2 &\leq 3 \\ 3x_1 + 2x_2 &\leq 4 \\ x_1, \; x_2 &\geq 0 \end{aligned}$$

1) maximize: $12x_1 + 3x_2 + 5x_3$

$$7x_1 + 4x_2 + 2x_3 \leq 100$$
$$2x_1 + 7x_3 \leq 50$$
$$x_1, x_2, x_3 \geq 0$$

minimize $100y_1 + 50y_2$

s.t. $7y_1 + 2y_2 \geq 12$
$$4y_1 \geq 3$$
$$2y_1 + 7y_2 \geq 5$$
$$y_1, y_2 \geq 0$$

2) maximize $2x_1 + 3x_2$

subject to:

$$4x_1 + 8x_2 \leq 12$$
$$2x_1 + x_2 \leq 3$$
$$3x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

minimize $12y_1 + 3y_2 + 4y_3$

s.t.

$$4y_1 + 2y_2 + 3y_3 \geq 2$$
$$8y_1 + y_2 + 2y_3 \geq 3$$
$$y_1, y_2, y_3 \geq 0$$

Primal:

$$\text{maximize} \quad c^T x$$
$$\text{subject to} \quad Ax \leq b$$
$$x \geq 0$$

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$y_1 \times \quad a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1$$
$$y_2 \times \quad \vdots$$
$$\vdots$$
$$y_m \times \quad a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m$$

$$\text{minimize} \quad b_1 y_1 + \cdots + b_m y_m$$

$$a_{11} y_1 + a_{21} y_2 + \cdots + a_{m1} y_m \geq c_1$$
$$\vdots \qquad \qquad \vdots$$
$$a_{1n} y_1 + a_{2n} y_2 + \cdots + a_{mn} y_m \geq c_n$$

Dual: $\quad \text{minimize} \quad b^T y$

$$\text{subject to} \quad A^T y \geq c$$
$$y \geq 0$$

Part 2: **Weak Duality Theorem**

Primal

maximize $c^T x$

subject to $Ax \le b$

$x \ge 0$

Dual

minimize $b^T y$

subject to $A^T y \ge c$

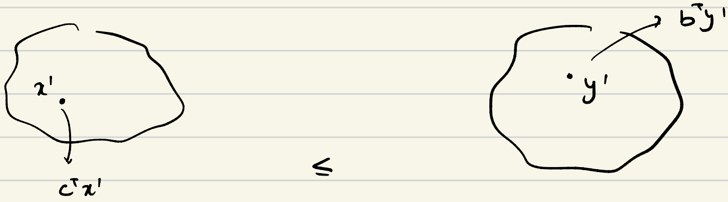
$y \ge 0$

For every feasible solution x' of Primal, and for every feasible solution y' of Dual, we have:

$$c^T x' \le b^T y'$$

Primal LP:

Dual LP



$$c^T x' = c_1 x_1' + c_2 x_2' + \dots + c_n x_n'$$

$$c_i \le a_{1i} y_1' + a_{2i} y_2' + \dots + a_{mi} y_m'$$

$$\sum_{i=1}^{n} c_i x_i' \le \sum_{i=1}^{n} (a_{1i} y_1' + \dots a_{mi} y_m')$$

$$c^T x' = c_1 x_1' + c_2 x_2' + \cdots + c_n x_n'$$

$$c_i \le a_{1i} y_1' + a_{2i} y_2' + \cdots + a_{mi} y_m'$$

$$c^T x' = \sum_{i=1}^{n} c_i x_i' \le \sum_{i=1}^{n} (a_{1i} y_1' + \cdots a_{mi} y_m') x_i'$$

Collecting terms belonging to some y_j

$$\le (a_{11} x_1' + a_{12} x_2' + \cdots + a_{1n} x_n') y_1'$$

$$+ \cdots$$

$$+ (a_{m1} x_1' + a_{m2} x_2' + \cdots + a_{mn} x_n') y_m'$$

$$\le b_1 y_1' + \cdots + b_m y_m'$$

$$= b^T y'$$

Corollary of Weak Duality:

1) If Primal is unbounded, dual is infeasible.

2) If Dual is unbounded from below, primal is infeasible.

1) Follows directly from weak duality.

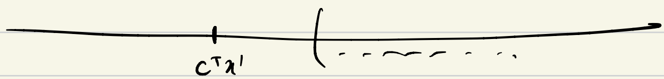
2) Consider the contra positive:

If primal is feasible, dual is bounded from below.

↳ follows from weak duality.

If x' is feasible for primal,

cost of dual is above $c^T x'$



Part 3: Writing dual for different LP forms

Primal	Dual
maximize $c^T x$	minimize $b^T y$
subject to $Ax \leq b$	subject to $A^T y \geq c$
$x \geq 0$	$y \geq 0$

1. $x_i \leq 0$

maximize $2x_1 + 3x_2$

subject to:

$$
\begin{aligned}
y_1 \times \quad 4x_1 + 8x_2 &\leq 12 \\
y_2 \times \quad 2x_1 + x_2 &\leq 3 \\
y_3 \times \quad 3x_1 + 2x_2 &\leq 4 \\
x_1 &\geq 0 \\
x_2 &\leq 0
\end{aligned}
$$

After multiplying with y_1, y_2, y_3:

We want: $2x_1 \leq (4y_1 + 2y_2 + 3y_3)\, x_1$

$$3x_2 \leq (8y_1 + y_2 + 2y_3)\, x_2$$

Since $x_1 \geq 0$, we achieve this by:

$$2 \leq 4y_1 + 2y_2 + 3y_3$$

But since $x_2 \leq 0$, we will achieve this by:

$$8y_1 + y_2 + 2y_3 \leq 3$$

maximize $\quad 2x_1 + 3x_2$

subject to:

$$
\begin{array}{llcl}
y_1 \times & 4x_1 + 8x_2 & \leq & 12 \\
y_2 \times & 2x_1 + x_2 & \leq & 3 \\
y_3 \times & 3x_1 + 2x_2 & \leq & 4 \\
 & x_1 & \geq & 0 \\
 & x_2 & \leq & 0
\end{array}
$$

minimize $\quad 12y_1 + 3y_2 + 4y_3$

s.t.

$$
\begin{array}{lcl}
4y_1 + 2y_2 + 3y_3 & \geq & 2 \\
8y_1 + y_2 + 2y_3 & \leq & 3 \\
y_1, y_2, y_3 & \geq & 0
\end{array}
$$

---

2) x_i is unconstrained.

maximize $\quad 2x_1 + 3x_2$

subject to:

$$
\begin{array}{llcl}
y_1 \times & 4x_1 + 8x_2 & \leq & 12 \\
y_2 \times & 2x_1 + x_2 & \leq & 3 \\
y_3 \times & 3x_1 + 2x_2 & \leq & 4 \\
 & x_1 & \geq & 0 \\
 & x_2 & & \text{unconstrained}
\end{array}
$$

Coeff. of x_2:

We want: $\quad 3x_2 \leq (8y_1 + y_2 + 2y_3)\, x_2$

$\rightarrow \quad 8y_1 + y_2 + 2y_3 \leq 3 \quad \times \quad$ will not work when $x_2 \geq 0$

$8y_1 + y_2 + 2y_3 \geq 3 \quad \times \quad$ will not work when $x_2 \leq 0$

$$8y_1 + y_2 + 2y_3 = 3$$

maximize $2x_1 + 3x_2$

subject to:

$$y_1 \times \quad 4x_1 + 8x_2 \leq 12$$
$$y_2 \times \quad 2x_1 + x_2 \leq 3$$
$$y_3 \times \quad 3x_1 + 2x_2 \leq 4$$
$$x_1 \geq 0$$
$$x_2 \text{ unconstrained}$$

minimize $12y_1 + 3y_2 + 4y_3$

s.t.

$$4y_1 + 2y_2 + 3y_3 \geq 2$$
$$8y_1 + y_2 + 2y_3 = 3$$
$$y_1, y_2, y_3 \geq 0$$

---

3. $A_i x_i \geq b_i$

maximize $2x_1 + 3x_2$

subject to:

$$y_1 \times \quad 4x_1 + 8x_2 \geq 12$$
$$y_2 \times \quad 2x_1 + x_2 \leq 3$$
$$y_3 \times \quad 3x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

minimize $12y_1 + 3y_2 + 4y_3$

$$4y_1 + 2y_2 + 3y_3 \geq 2$$
$$8y_1 + y_2 + 2y_3 \geq 3$$
$$y_1 \leq 0$$
$$y_2, y_3 \geq 0$$

---

4) $A_i x_i = b_i$

maximize $2x_1 + 3x_2$

subject to:

$$y_1 \times \quad 4x_1 + 8x_2 = 12$$
$$y_2 \times \quad 2x_1 + x_2 \leq 3$$
$$y_3 \times \quad 3x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

minimize $12y_1 + 3y_2 + 4y_3$

$$4y_1 + 2y_2 + 3y_3 \geq 2$$
$$8y_1 + y_2 + 2y_3 \geq 3$$
$$y_1 \text{ unconstrained}$$
$$y_2, y_3 \geq 0$$

Primal

$$
\begin{aligned}
&\text{maximize} && c^T x \\
\text{s.t.} \quad & && Ax \le b \\
& && x \ge 0
\end{aligned}
$$

$x_i \le 0$

$A^i y \le c_i$

x_i unconst.

$A^i y = c_i$

$A_i x \ge b_i$

$y_i \le 0$

$A_i x = b_i$

y_i unconst.

Dual

$$
\begin{aligned}
&\text{minimize} && b^T y \\
\text{s.t.} \quad & && A^T y \ge c \\
& && y \ge 0
\end{aligned}
$$

Primal:

Dual

minimize $b^T y$

s.t. $A^T y \geq c$

$y \geq 0$

$\longrightarrow$

maximize $c^T x$

s.t. $Ax \leq b$

$x \leq 0$

If we had started with the above primal, and we wanted a dual LP that can give a lower bound to the optimum, we will get the LP on the right.

→ We call these two LPs as primal - dual pairs.

Write duals:

1) maximize $5x_1 + 3x_2$

$$
\begin{aligned}
\text{s.t.} \quad & 4x_1 + 2x_2 \leq 4 \\
& x_1 - 3x_2 \geq 3 \\
& 5x_1 + 2x_2 = 2 \\
\\
& x_1 \geq 0 \\
& x_2 \ \text{unconst.}
\end{aligned}
$$

2) minimize $2y_1 + 4y_2 + 3y_3$

$$
\begin{aligned}
\text{s.t.} \quad & 7y_1 + 3y_2 + 4y_3 \geq 4 \\
& 2y_1 + y_2 = 2 \\
& y_1 + y_2 + y_3 \leq 3 \\
\\
& y_1 \leq 0 \\
& y_2 \geq 0 \\
& y_2 \ \text{unconstr.}
\end{aligned}
$$

1) maximize $5x_1 + 3x_2$

s.t.

$$4x_1 + 2x_2 \leq 4$$
$$x_1 - 3x_2 \geq 3$$
$$5x_1 + 2x_2 = 2$$

$$x_1 \geq 0$$
x_2 unconst.

minimize $4y_1 + 3y_2 + 2y_3$

$$4y_1 + y_2 + 5y_3 \geq 5$$
$$2y_1 - 3y_2 + 2y_2 = 3$$

$y_1 \geq 0$, $y_2 \leq 0$,
y_3 unconstr.

2) minimize $2y_1 + 4y_2 + 3y_3$

s.t.

$$7y_1 + 3y_2 + 4y_3 \geq 4$$
$$2y_1 + y_2 = 2$$
$$y_1 + y_2 + y_3 \leq 3$$

$$y_1 \leq 0$$
$$y_2 \geq 0$$
y_2 unconstr.

maximize $4x_1 + 2x_2 + 3x_3$

$$7x_1 + 2x_2 + x_3 \geq 2$$
$$3x_1 + x_2 + x_3 \leq 4$$
$$4x_1 + x_3 = 3$$

$$x_1 \geq 0$$
x_2 unconstr.
$$x_3 \leq 0$$

Part 4: DUALITY THEOREM

Primal (P)	Dual (D)
maximize $c^T x$	minimize $b^T y$
s.t. $Ax \leq b$	s.t. $A^T y \geq c$
$x \geq 0$	$y \geq 0$

Only the following can occur:

- 1. Either both (P) and (D) are infeasible.
- 2. (P) is unbounded and (D) is infeasible.
- 3. (P) is infeasible and (D) is unbounded.
- 4. Both (P) and (D) have optima, and optimal cost (P) = optimal cost (D).

comes from strong duality

(P) \ (D)	infeasible	bounded	unbounded
infeasible	✓	✗ (S)	✓
bounded	✗ (S)	✓	✗ (W)
unbounded	✓	✗ (W)	✗ (W)

Summary.

- 1. Primal-dual pairs, example
- 2. Weak duality thm. + proof
- 3. (Strong) duality thm statement.