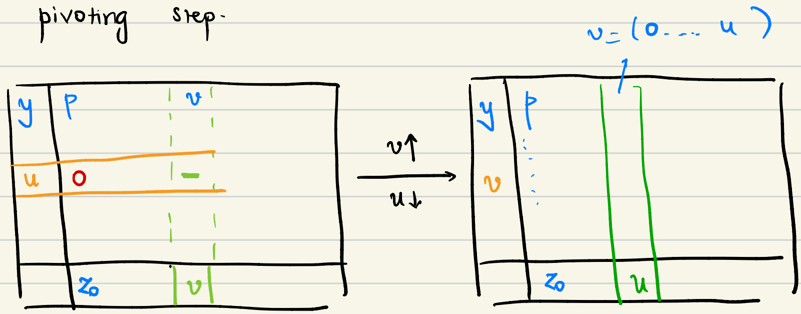
LINEAR PROGRAMMING & COMBINATORIAL OPTIMIZATION

LECTURE 7

Bland's pivoting rule.

Recall:

degenerate pivoting step.



- BFS does not change
- cost does not change
- only the basis changes.

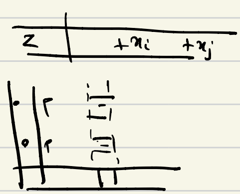
Cycling: This degenerate pivoting could lead to a cycle of tableaus.

Why does degeneracy appear?

→ Multiple bases give a single bfs

Bland's rule: Variables: $\{x_1, x_2, \dots\ x_n\}$

Pivoting step: Which variable enters?
Which variable leaves?



When there is choice, pick the variable with the least index.
(in entry or exit)

Example:

Maximize $\qquad x_1 - 2x_2 + x_3$

subject to

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &\leq 12 \\
2x_1 + x_2 - x_3 &\leq 6 \\
-x_1 + 3x_2 &\leq 9 \\
x_1,\ x_2 &\geq 0
\end{aligned}
$$

$$
\begin{array}{rcl}
x_4 &=& 12 - x_1 - 2x_2 - x_3 \\
x_5 &=& 6 - 2x_1 - x_2 + x_3 \\
x_6 &=& 9 + x_1 - 3x_2 \\
\hline
z &=& x_1 - 2x_2 + x_3
\end{array}
$$

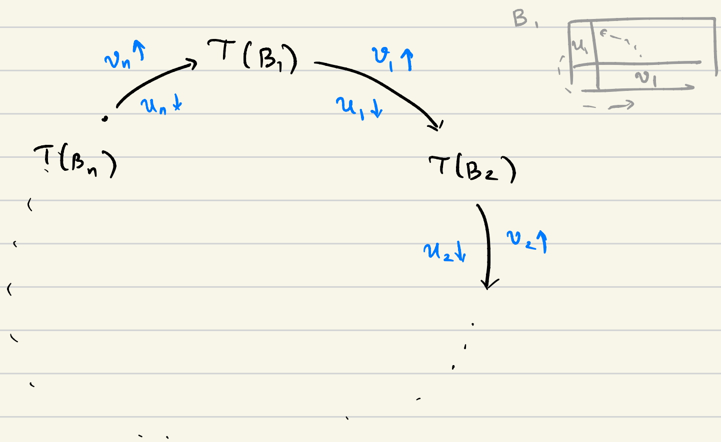
Bland's rule says choose x_1.

$x_1 \uparrow \ \vdots \ x_5 \downarrow$

. - - -

Theorem: The simplex method with Bland's rule terminates.

Proof: Suppose there is a cycle.



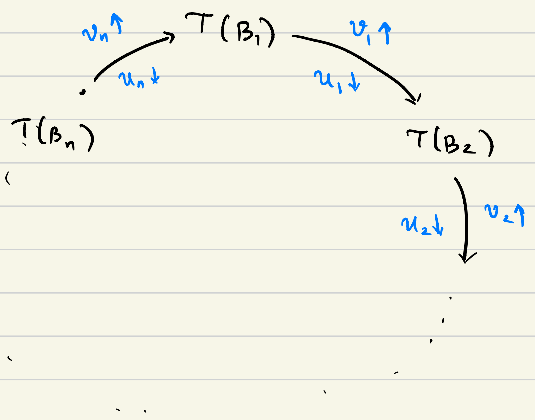
Some observations:

1. $\{v_1, v_2, \ldots, v_n\} = \{u_1, u_2, \ldots, u_n\}$

Due to the cycle, any variable that leaves the basis should renter.

$$F = \{v_1, v_2, \ldots v_n\} = \{u_1, u_2, \ldots, u_n\}$$

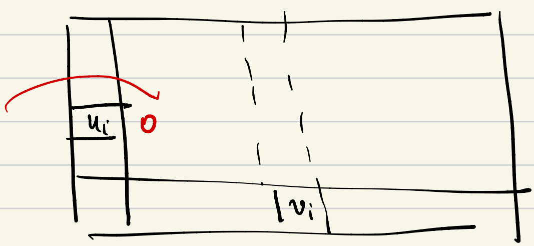
2. The BFS in the entire cycle is the same.



Cost remains the same in each pivoting step.

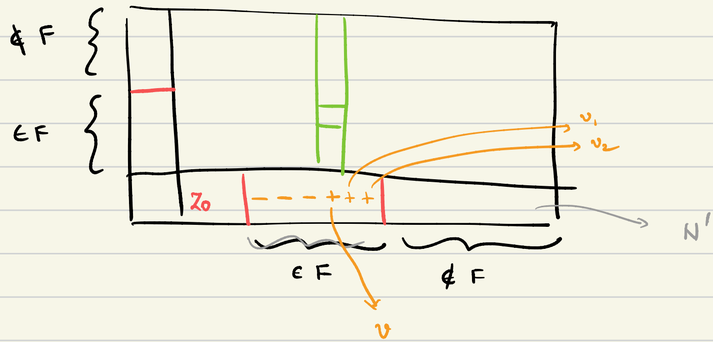
→ This will imply that :

This constant is 0



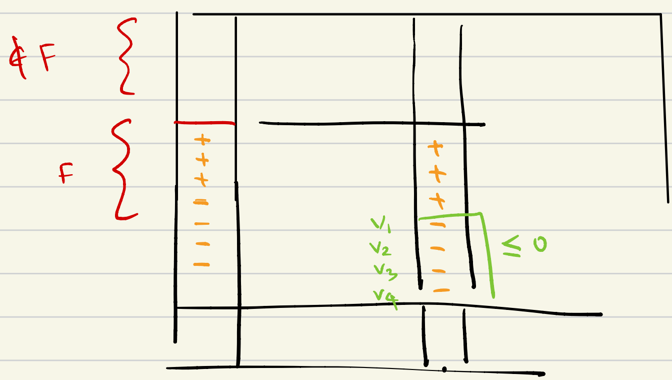
Therefore, the bfs does not change.

Take some arbitrary tableau in the cycle.



If v_1, v_2 are fixed at 0, this tableau gives a proof that the cost is bounded.

For the LP with added constraints $v_1 = v_2 = 0$, and all variables in N' made to 0.
z_0 is the optimum.

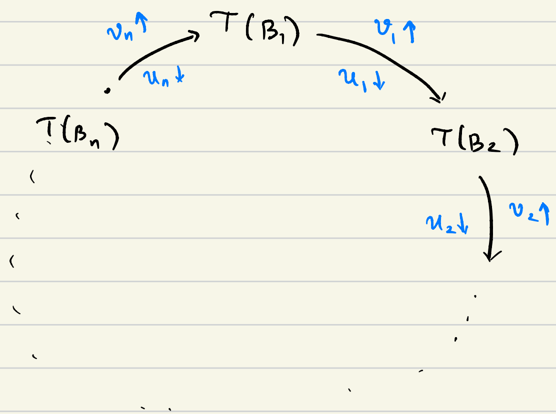


Suppose we change the LP : $v_1, v_2, v_3, v_4 \leq 0$

The above tableau is proof that the LP is unbounded.

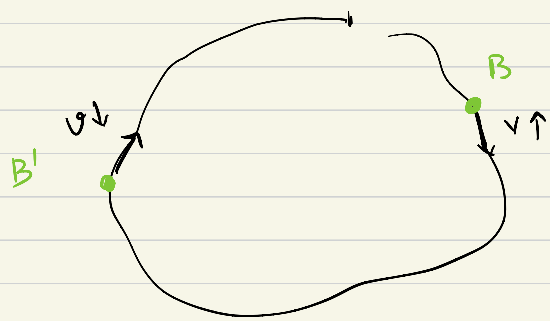
We will now choose two tableaus in the cycle for which the LPs from the previous slide will be the same.

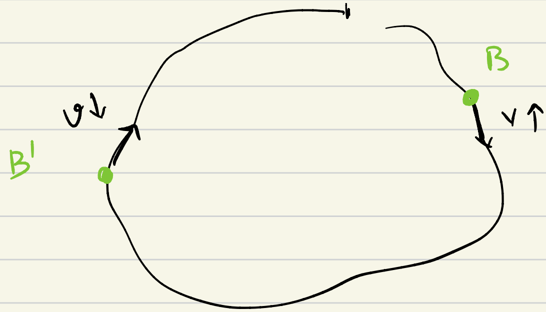
→ This will be a contradiction because one tableau says that the LP is bounded and the other prove that LP is unbounded.



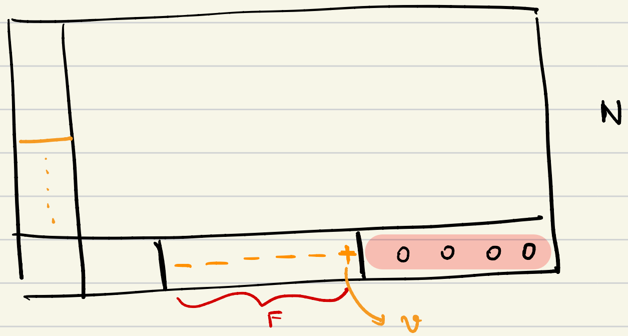
→ $F = \{ v_1, v_2, \ldots v_n \}$

Let v be the variable with the highest index in F.





T(B):



Consider a tableau from where v enters the basis.

Let B: basic variables and N: non-basic var.

LP:

maximize $c^T x$

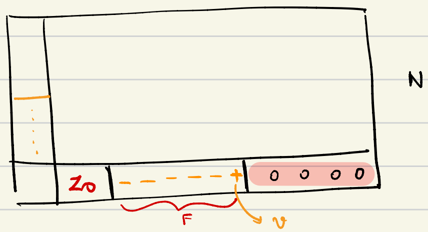
$$Ax = b$$

$$y = 0 \quad \forall\, y \in N \setminus F$$

$$v \leq 0$$

$$y \geq 0 \quad \forall\, y \in F \setminus \{v\}$$

$T(B)$:



New LP:

maximize $c^T x$

$$Ax = b$$

$$y = 0 \quad \forall\, y \in N \setminus F$$

$$v \leq 0$$

$$y \geq 0 \quad \forall\, y \in F \setminus \{v\}$$

1. $T(B)$ is a tableau for this new LP as well.

   → it is a basis due to $Ax = b$

   → the new inequalities are also satisfied by putting all variables in N to 0.

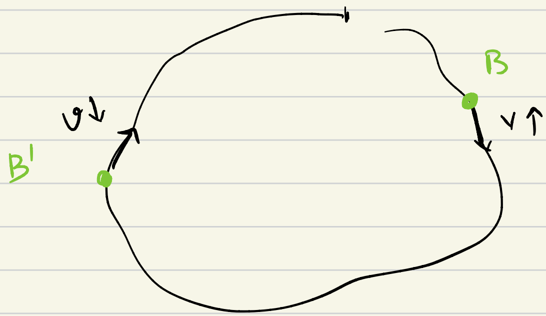
   → So it's a feasible basis for new LP.

2. The cost depends negatively for variables $y \in F \setminus \{v\}$

   → Increasing y will decrease cost.

   v can be increased only upto 0.

This implies that z_0 is optimum for the new LP.



Let us look at B'.

In the column corresponding to u, the coefficients of variables in $B' \cap F$ will be ≥ 0 except coeff. of v which will be < 0

New LP:

maximize $c^T x$

$Ax = b$

$y = 0 \quad \forall\, y \in N \setminus F$

$v \leq 0$

$y \geq 0 \quad \forall\, y \in F \setminus \{v\}$

T(B') proves that the new LP is unbounded.

- Notice that $B' \setminus F = B \setminus F$

  $N' \setminus F = N \setminus F$

- Let u be the variable that enters the basis from B'.

Keeping all variables in $N' \setminus \{u\}$ to be 0 and increasing u arbitrarily gives feasible solutions for the new LP, with increasing cost.

New LP:

maximize $c^T x$

$Ax = b$

$y = 0 \quad \forall y \in N \setminus F$

$v \leq 0$

$y \geq 0 \quad \forall y \in F \setminus \{v\}$

- no constraint on $B \setminus F$
- constraint $y \geq 0$ $y \in F \setminus \{v\}$ hold when increasing u.
- $y = 0 \ \forall y \in N \setminus F$ holds
- $v \leq 0$ holds.

From the cycle, we have two tableaus B and B'

s.t. T(B) proves new LP has an optimum

T(B') proves new LP is unbounded.

- Contradiction

- Hence there can be no cycle.

- Simplex terminates.