LINEAR PROGRAMMING & COMBINATORIAL OPTIMIZATION

LECTURE 6

Plan for today's lecture:

- 1. Proof of correctness of simplex (assuming termination)
- 2. A geometric interpretation of simplex
- 3. Solutions to Problem Sheet 2.

Proof of correctness of the simplex method:

maximize $c^T x$

subj to $Ax = b$ A: $m \times n$, rank m.

$x \geqslant 0$

Feasible basis: $B \subseteq \{1, 2, \ldots, n\}$

$B = \{i_1, i_2, \ldots, i_m\}$ s.t.

$A^{i_1}\ A^{i_2} \ldots A^{i_m}$ are linearly indep.

- $\exists$ a feasible soln. where $x_j = 0$ $\forall j \notin B$.

Simplex:

$B_0 \longrightarrow B_1 \longrightarrow \ldots\ldots \longrightarrow$

initial feasible basis next basis . . .

feasible	B		$B' = B \setminus \{u\} \cup \{v\}$
Tableau	$T(B)$	$\xrightarrow[\text{Pivot}]{v\uparrow,\ u\downarrow}$	$T(B')$
bfs corresponding to B	$x(B)$		$x(B')$

To show:

-1. Each feasible basis corresponds to a unique tableau $T(B)$ ✓
(last lecture)

-2. Each feasible basis gives a unique bfs. This is obtained by setting all non-basic variables to 0, and deriving values for basic variables from the tableau. ✓

-3. B' is feasible.

-4. Cost at $x(B')$ is $\geq$ cost at $x(B)$

Suppose simplex terminates at B.

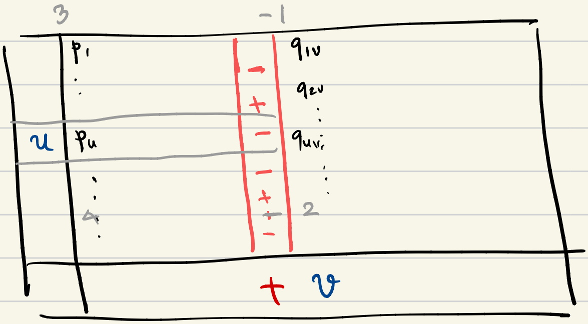
- if all coefficients of variables in the cost row are ≤ 0, then optimum is attained. ✓

- when simplex terminates saying unbounded, LP is indeed unbdd.

feasible	B		$B' = B \setminus \{u\} \cup \{v\}$
Tableau	$T(B)$	$\xrightarrow[\text{Pivot}]{v\uparrow,\ u\downarrow}$	$T(B')$
bfs corresponding to B	$x(B)$		$x(B')$

B′ is a feasible basis.

$T(B)$



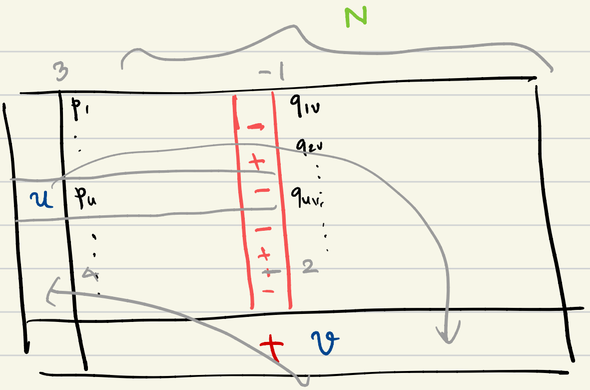
−1. Coefficient of v in cost row is positive.

−2. In the column corresponding to v, there are negative coefficients.

$$\min \left\{ -\frac{p_j}{q_{jv}} \;\middle|\; j \text{ s.t. } q_{jv} < 0 \right\}$$

u is the variable which gives the minimum value for the above quantity.

T(B)



$B' = B \setminus \{u\} \cup \{v\}$ is feasible.

$$B = \begin{array}{c} \\ \\ u \\ \\ \end{array}\!\!\begin{array}{cccc} & & u & \\ \end{array}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \vdots & \vdots \\ \vdots & \ddots & 1 & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix} \qquad N = \begin{array}{c} \\ u = \\ \\ \end{array}\begin{bmatrix} q_{11} & \overset{v}{\cdots} & q_{1n-m} \\ q_{12} & q_{uv} & \vdots \\ \vdots & & q_{m,n-m} \end{bmatrix}$$

$I^{m \times m}$ $\qquad\qquad$ $m \times (n-m)$

$q_{uv} \neq 0$

$$B' = \begin{array}{c} \\ \\ u \\ \\ \end{array}\begin{bmatrix} 1 & 0 & \overset{v}{} & 0 \\ 0 & 1 & & \vdots \\ 0 & 0 & q_{uv} & 0 \quad 0 \quad \vdots \\ 0 & 0 & & 1 \end{bmatrix}$$

In the u^{th} row, the only non-zero entry is q_{uv}

This shows that columns of B' are linearly independent.

$q_{uv} \neq 0$

$$B' = \begin{array}{cccccccc} & & & v & & & \\ & 1 & 0 & & & & 0 \\ & 0 & 1 & & & & \vdots \\ & \vdots & \vdots & & & & \\ u & 0 & 0 & q_{uv} & 0 & 0 & \vdots \\ & 0 & 0 & & & & 1 \end{array}$$

In the u^{th} row, the only non-zero entry is q_{uv}

This shows that columns of B' are linearly independent.

- It can be shown that the columns corresponding to indices B' in the original matrix that we started off with are also linearly independent.

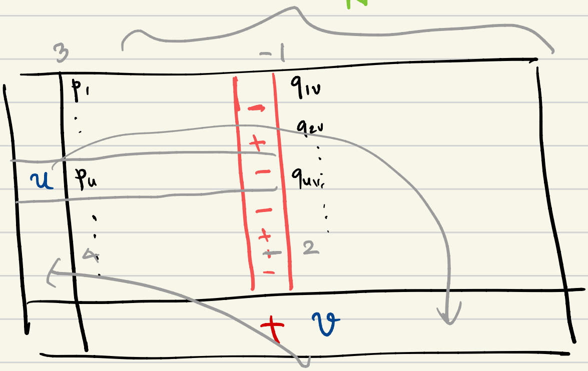
(Exercise)

- By our choice of 'u', we can also see that the resulting basis B' also gives a feasible soln.

↳ B' is feasible.

cost at $x(B')$ is $\geq$ cost at $x(B)$

$T(B)$



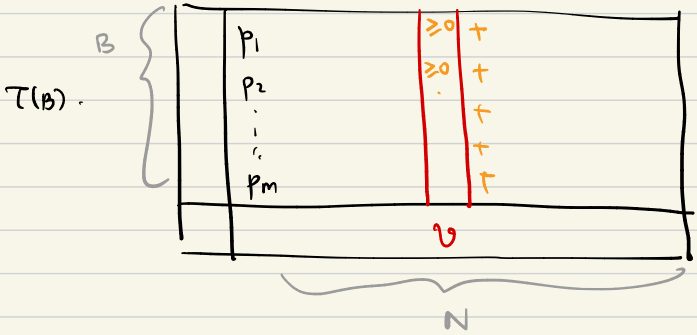
→ In the resulting tableau, all variables of N except v are 0. v has been increased.

By looking at cost vector in B;

$$z = z_0 + \underbrace{\cdots}_{0} + \underbrace{a_v}_{\substack{=0 \text{ in } B \\ \geq 0 \text{ in } B'}} \cdot v + \underbrace{\cdots}_{0}$$

∴ cost at B' is bigger than cost at B.

When simplex returns "unbounded", LP is indeed unbounded.



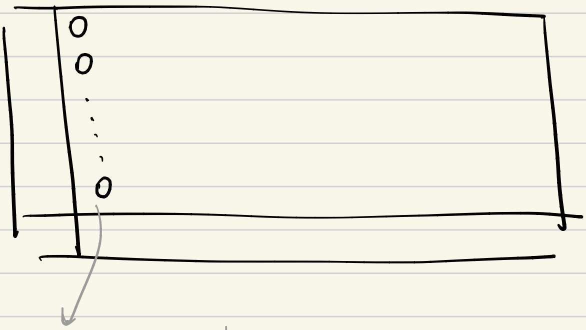
$x(B)$: call it x.

The tableau gives a $w \in \mathbb{R}^n$ s.t. $Aw = 0$

$c^T w > 0$

w: Set $w(v) = 1$

$w(j) = 0 \quad \forall j \in N \setminus \{v\}$



change all p_i's to 0.

Keep rest the same. $\Big\}$ $Ax = 0$

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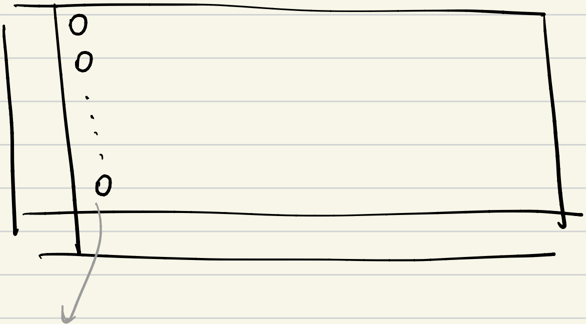


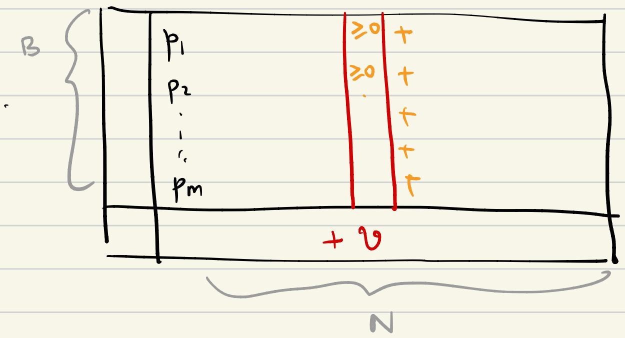
change all p_i's to 0. } $Ax = 0$
Keep rest the same.

For all $j \in B$, substitute $v = 1$, and other non-basic variables to 0

and derive value from the above modified tableau.

$$\begin{array}{rcl} x_1 &=& 5 - x_3 + x_4 \\ x_2 &=& 2 - 2x_3 + 2x_4 \\ \hline z &=& 4 + x_3 + x_4 \end{array} \quad \Bigg| \quad \begin{array}{rcl} x_1 &=& 0 - x_3 + x_4 \\ x_2 &=& 0 - 2x_3 + 2x_4 \\ \hline z &=& 4 + x_3 + x_4 \end{array}$$

$$w = \begin{array}{c} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \end{array} \begin{array}{l} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}$$



In the w that we get from the above tableau:

$$w(j) \geq 0 \quad \forall j.$$

- $x(B) + \lambda w$ satisfies $Ax = b$
  $x \geq 0$

- $c^T w = r_v$: (coefficient of v in the cost row of B)

  - $r_v > 0$.

  ∴ $c^T w > 0$

- $c^T(x(B) + \lambda w) = c^T x(B) + \lambda r_v$

  → Increasing λ gives feasible points with increasing cost

  ⇒ LP is unbounded.

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To show:

-1. Each feasible basis corresponds to a unique tableau $T(B)$ ✓

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-2. Each feasible basis gives a unique bfs. This is obtained by setting all non-basic variables to 0, and deriving values for basic variables from the tableau. ✓

✓ -3. B' is feasible.

✓ -4. Cost at $x(B')$ is $\geq$ cost at $x(B)$

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✓ – when simplex terminates saying unbounded, LP is indeed unbdd.

Geometric interpretation:

Consider $\mathbb{R}^n$

Closed half-space: $\{ x \in \mathbb{R}^n \mid a_1 x_1 + a_2 x_2 + \dots + a_n x_n \le b \}$

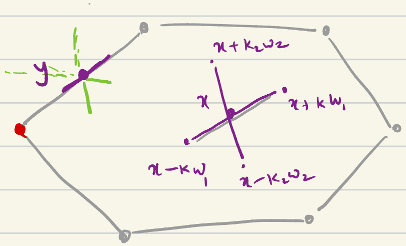
Convex polyhedron: Intersection of finitely many closed half-spaces.

Feasible region of Equational form results in a convex polyhedron.

Interior point: a point x s.t. $\exists w \in \mathbb{R}^n$ and $k \in \mathbb{R}$

, $x + \lambda w$ is in the convex polyhedron for all $\lambda \in [-k, k]$

Vertex: a point in the polyhedron that is not an interior point.



Edge: x belongs to an edge if it is an interior point s.t. there is is exactly one $w \in \mathbb{R}^n$ (modulo linear dependence) and some $k \in \mathbb{R}$ s.t.

$x + \lambda w$ is in the polyhedron for all $\lambda \in [-k, +k]$

Edge: An edge is a of feasible points.

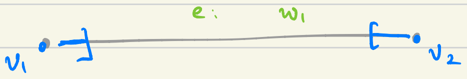
- To every edge, we can associate a unique "w" coming from the definition. This is the direction along which we can move to stay inside the edge.

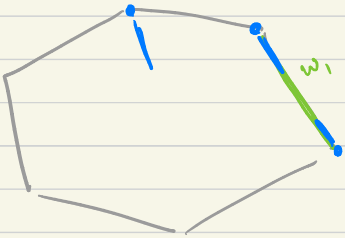
A vertex v is a corner point of edge e (with associated w)

if $v + \lambda w$ is in 'e' for either $\lambda \in [0, .. k]$

or $\lambda \in [-k, 0]$

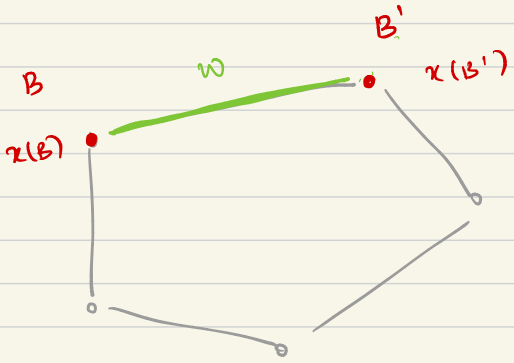
for some $k \in \mathbb{R}$



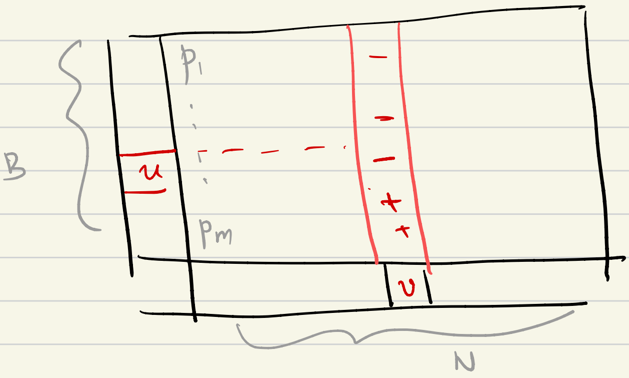


Adjacent vertices: Vertices that are corner points of an edge.

Simplex method: Each pivot step moves to an adjacent vertex.



$T(B)$:



Consider the tableau where all p_i are replaced with 0. + $T^0(B)$

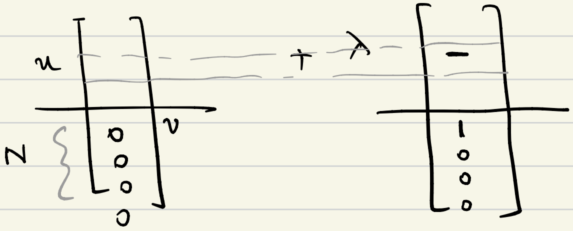
w: $w(v) = 1$

$w(j) = 0 \quad \forall\ j \in N \setminus \{v\}$

$\forall j \in B$: $w(j)$ = value obtained by above values to N, in tableau $T^0(B)$

− $Aw = 0$, $c^T w > 0$

$x(B) + \lambda w$



$x(B')$: consider the biggest value for λ that keeps $x(B) + \lambda w$ feasible.

This will correspond to the variable that is removed from the basis in the pivoting step.

$x(B) =$

$x(B') = x(B) + \lambda^* w$

$E = \{ x(B) + \lambda w \mid 0 < \lambda < \lambda^* \}$

↳ is an edge.

→ $y \in E$. We can move around "w".

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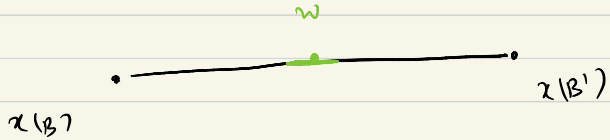
↳ is an edge.

→ $y \in E$. We can move around "w".

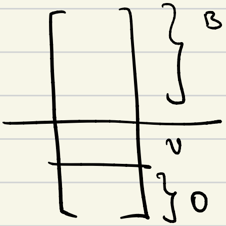
$y = x(B) + \lambda_1 w$.

$\hookrightarrow 0 < \lambda_1 < \lambda^*$

Take k to be some value $< \min(\lambda_1, \lambda^* - \lambda_1)$



– For any $y \in E$.



Value of $\underline{N \setminus \{v\}} = 0$

∴ For any other w' which is a potential candidate to move around, we require

$$w'(j) = 0 \quad \forall j \in N \setminus \{v\}$$

$$w' = \left[\begin{array}{c} \\ \\ \\ \hline k \\ \hline 0 \\ \vdots \\ 0 \end{array}\right] \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} B \\ v \\ \left.\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}\right\} N \setminus \{v\} \end{array}$$

But then; fixing a value for v derives the values for variables in B.

From tableau $T^{0}(B)$

Since we need $Aw' = 0$

This implies that w (from the edge) and w' are linearly dependent.

$\therefore$ $E = \{ x(B) + \lambda w \mid 0 < \lambda < \lambda^* \}$ form an edge.

→ It is also easy to see that $x(B)$ and $x(B')$ are corner points of this edge E.

$\therefore$ $x(B')$ is adjacent to $x(B)$.

Summary:

- 1. Proof of correctness of simplex

- 2. Geometric interpretation: simplex starts at a bfs, and moves to an adjacent bfs with "better" cost.