LINEAR PROGRAMMING & COMBINATORIAL OPTIMIZATION

LECTURE 5

Last lecture: Simplex method, examples

Today's lecture:

- More examples
  - degeneracy
  - finding initial feasible basis

- Formalizing the simplex method

Example:

maximize x_2

subject to:

$$\begin{aligned} -x_1 + x_2 &\le 0 \\ x_1 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

$$\begin{array}{lcl} x_3 &=& 0 + x_1 - x_2 \\ x_4 &=& 2 - x_1 \\ \hline z &=& x_2 \end{array} \xrightarrow[x_3 \downarrow]{x_2 \uparrow} \begin{array}{lcl} x_2 &=& 0 + x_1 - x_3 \\ x_4 &=& 2 - x_1 \\ \hline z &=& 0 + x_1 - x_3 \end{array}$$

$\langle 0, 0, 0, 2 \rangle$

$B_1 = \{3, 4\}$

$B_2 = \{2, 4\}$

$\langle 0, 0, 0, 2 \rangle$

$x_4 \downarrow \quad x_1 \uparrow$

$$\begin{array}{lcl} x_2 &=& 2 - x_4 - x_3 \\ x_1 &=& 2 - x_4 \\ \hline z &=& \boxed{2} - x_4 - x_3 \end{array}$$

optimum

- The bfs $\langle 0, 0, 0, 2 \rangle$ corresponds to two different bases:

$B_1 = \{3, 4\}$ | $B_2 = \{2, 4\}$
$N_1 = \{1, 2\}$ | $N_2 = \{1, 3\}$

$\Rightarrow x_1 = 0$, $x_2 = 0$ | $\Rightarrow x_1 = 0$, $x_3 = 0$

- In the first pivoting step above, the bfs did not change. Hence the cost did not change.

Degeneracy in equational form:

An LP in equational form is said to be degenerate if several feasible bases correspond to a single bfs.

- In such a case, at least one of the basic variables in the bfs has to be 0.

- We have seen an instance of this in the previous example.

Degenerate pivot step:

- A pivot step in which basis changes, but the cost does not change.

Cycling:

- a sequence of degenerate pivot steps that brings back to a previous tableau.

Later, we will see some pivoting rules that will prevent cycling.

Example: Apply simplex on the following LP.

Maximize $x_1 + 2x_2$

Subject to:

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_2 + x_3 &= 2 \\ x_1, \; x_2, \; x_3 &\geq 0 \end{aligned}$$

How to start the simplex procedure?

↳ Find an initial tableau (need to find a bfs)

Finding an initial feasible basis:

- This is a hard problem. Finding a basis is easy. But we need to ensure that it is feasible. A brute force enumeration of all bases is not useful, since it is expensive and as difficult as finding the optimum.

- We will make use of an auxiliary LP to detect feasibility and infer a feasible basis

Back to example:

Maximize $x_1 + 2x_2$

Subject to:

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_2 + x_3 &= 2 \\ x_1,\ x_2,\ x_3 &\geq 0 \end{aligned}$$

- Add auxiliary variables x_4, x_5:

- Change constraints to:

$$\left.\begin{aligned} x_4 &= 4 - x_1 - 3x_2 - 3x_3 \\ x_5 &= 2 - 2x_2 - x_3 \end{aligned}\right\}$$

Find a feasible soln. to this system of inequalities s.t. $x_4 = x_5 = 0$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

- Consider objective maximize $-x_4 - x_5$

Maximize $x_1 + 2x_2$

Subject to:

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_2 + x_3 &= 2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

LP

$\rightarrow$

$$\boxed{\begin{array}{l} \text{Maximize } -x_4 - x_5 \\ \text{subject to:} \\ x_4 = 4 - x_1 - 3x_2 - x_3 \\ x_5 = 2 - 2x_2 - x_3 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}}$$

Auxiliary LP

Lemma: LP is feasible iff optimum of auxiliary LP is 0.

Proof: Notice that the objective of auxiliary LP ≤ 0.

- A feasible point of LP gives a point in aux. LP with cost 0.
- An optimum of aux. LP with cost 0 has $x_4 = x_5 = 0$. Hence the projection of optimum onto $\langle x_1, x_2, x_3 \rangle$ gives a feasible point of LP.

What is the advantage of auxiliary LP?

- A feasible basis can be detected easily. It is given by the extra variables.

For the above aux. LP

$$\begin{array}{ccccc} & & & & \\ \left[\begin{array}{ccccc} 1 & 3 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array}$$

$B = \{4, 5\}$ is a basis. Moreover, substituting $x_1, x_2, x_3 = 0$ gives $x_4 = 4$, $x_5 = 2$. $\rightarrow$ a bfs.

$\langle 0, 0, 0, 4, 2 \rangle$

Aux. LP.

$x_1, x_2, \ldots x_n$

$x_{n+1}, \ldots\ldots x_{n+m}$

Claim: -1) Simplex is easy to start for the aux. LP.

-2) Aux LP has an optimum.

1) $\langle x_{n+1}, \ldots\ldots x_{n+m} \rangle$ forms a feasible basis for the aux. LP.

$$x_1 + 2x_2 + x_3 = -2$$
$$x_2 + 5x_3 = 4$$

$$\begin{array}{rcrrrr} -2 & & & 0 & 0 & 0 \\ x_4 & = & -2 & -x_1 & -2x_2 & -x_3 \\ 4 \quad x_5 & = & 4 & & -x_2 & -5x_3 \end{array}$$

$$-x_1 - 2x_2 - x_3 = 2$$
$$x_2 + 5x_3 = 4$$

- 2) Aux LP has an optimum.

Cost function: $-x_{n+1} - x_{n+2} \cdots - x_{n+m}$

$$\text{cost} \leq 0$$

What is the advantage of auxiliary LP ?

- A feasible basis can be detected easily. It is given by the extra variables.

For the above aux. LP

$$\begin{array}{ccccc} & & & & \\ \left[\begin{matrix} 1 & 3 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \end{matrix}\right] \\ x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \end{array}$$

$B = \{4, 5\}$ is a basis. Moreover, substituting $x_1, x_2, x_3 = 0$ gives $x_4 = 4$, $x_5 = 2$. $\rightarrow$ a bfs.
$\langle 0, 0, 0, 4, 2 \rangle$

- Secondly, the cost of the aux LP ≤ 0. Therefore the optimum is attained at a bfs. The simplex method on the aux LP. gives a bfs for the aux LP.

- Suppose y^* is the bfs of aux LP that gives optimum

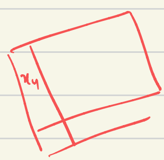
Basic variables of y^* do not include aux. variables

$\downarrow$

Same basis is feasible in LP

Some aux. variable is basic in the final tableau

$\downarrow$



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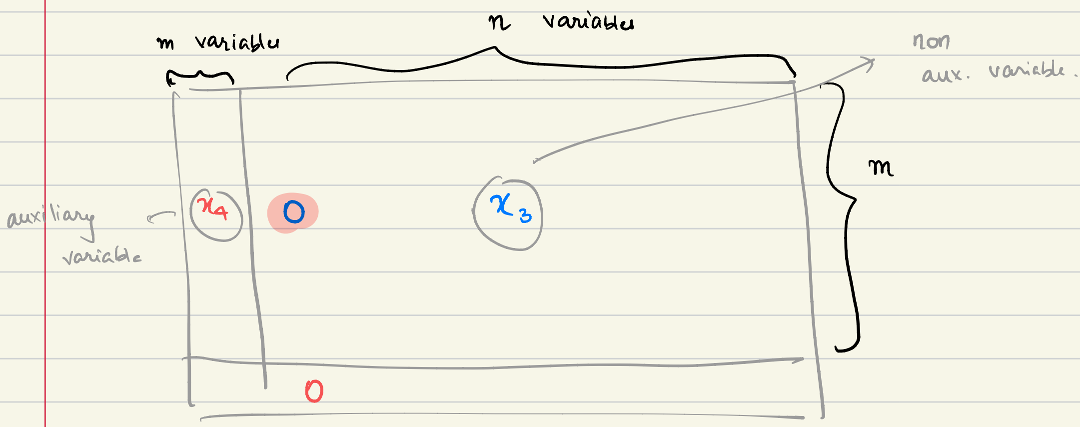
Final tableau of aux. LP.



→ In the final tableau, the constant corresponding to rows where the aux. variables are on the LHS will be 0.

→ We can rearrange the tableau to bring all aux variables to the RHS.

This will be a sequence of degenerate steps until all basic variables are among the original variables.

- Suppose y^* is the bfs of aux. LP that gives optimum

Basic variables of y^* do not include aux. variables

Same basis is feasible in LP

Some aux. variable is basic in the final tableau

This is a degenerate case where the aux. variable which is basic also has value 0. The tableau can be rewritten so that the original variables are on the LHS and aux. variables are on the RHS.

- without changing the bfs.

Example:

Maximize $x_1 + 2x_2$

Subject to:

$$x_1 + 3x_2 + x_3 = 4$$
$$2x_2 + x_3 = 2$$
$$x_1, x_2, x_3 \geq 0$$

LP

$\rightarrow$

Maximize $-x_4 - x_5$

subject to:

$$x_4 = 4 - x_1 - 3x_2 - x_3$$
$$x_5 = 2 - 2x_2 - x_3$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Auxiliary LP

Solution to auxiliary LP:

$$\begin{aligned} x_4 &= 4 - x_1 - 3x_2 - x_3 \\ x_5 &= 2 - 2x_2 - x_3 \\ \hline z &= \underbrace{-6 + x_1 + 5x_2 + 2x_3}_{-x_4 - x_5} \end{aligned} \xrightarrow[x_4 \downarrow]{x_1 \uparrow} \begin{aligned} x_1 &= 4 - x_4 - 3x_2 - x_3 \\ x_5 &= 2 - 2x_2 - x_3 \\ \hline z &= -2 - x_4 + 2x_2 + x_3 \end{aligned}$$

$$\Big\downarrow \; x_5 \downarrow \quad x_3 \uparrow$$

Optimum!

$$\begin{aligned} x_1 &= 2 - x_4 - x_2 - x_5 \\ x_3 &= 2 - 2x_2 - x_5 \\ \hline z &= -x_4 - x_5 \end{aligned}$$

Optimum of aux. LP is 0 with basis $\{1, 3\}$

Notice that $\{1,3\}$ is a feasible basis of original LP too.

Maximize $x_1 + 2x_2$

Subject to:

$$x_1 + 3x_2 + x_3 = 4$$
$$2x_2 + x_3 = 2$$
$$x_1, \; x_2, \; x_3 \geq 0$$

LP

$$x_1 + x_3 = 4 - 3x_2$$
$$\boxed{\begin{aligned} x_3 &= 2 - 2x_2 \\ x_1 &= 2 - x_2 \end{aligned}}$$

We have inferred that $\{1, 3\}$ is a feasible basis.

Projecting the final tableau of the aux. LP to $\{x_1, x_2, x_3\}$ gives the initial tableau for LP:

$$\begin{array}{rcl} x_1 &=& 2 - x_2 \\ x_3 &=& 2 - 2x_2 \\ \hline z &=& 2 + x_2 \end{array} \xrightarrow[x_3 \downarrow]{x_2 \uparrow} \begin{array}{rcl} x_1 &=& 1 + x_3/2 \\ x_2 &=& 1 - x_3/2 \\ \hline z &=& 3 - x_3/2 \end{array}$$

Optimum

bfs : $\langle 1, 1, 0 \rangle$ gives optimum

Cost : 3

Formalization of the simplex method:

At each step there is a bfs represented as a tableau.

either stop with an answer or go to next step

$$\begin{tikzcd} B \arrow[r, leftrightarrow] \arrow[d] & T(B) \arrow[d] \\ B' \arrow[r, leftrightarrow] & T(B') \end{tikzcd}$$

Tableau of B

Tableau of a bfs:

A simplex tableau determined by a feasible basis B is a system of $m+1$ linear equations in variables

$$x_1, x_2, \ldots, x_n, z$$

that has the same set of solutions as $Ax = b,\ x \geq 0$ and in matrix notation looks like:

$$\begin{bmatrix} x_{i_1} \\ x_{i_2} \\ \vdots \\ x_{i_m} \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix} +$$

$$\begin{array}{l} x_B = p + Q x_N \\ \hline z = z_0 + r^T x_N \end{array}$$

$x_B \in \mathbb{R}^m$, $x_N \in \mathbb{R}^{n-m}$, $p \geq 0$ and $p \in \mathbb{R}^m$, Q is an $m \times (n-m)$ matrix

$z_0 \in \mathbb{R}$, $r \in \mathbb{R}^{n-m}$

$$\underbrace{\begin{bmatrix} x_{i_1} \\ x_{i_2} \\ \vdots \\ x_{i_m} \end{bmatrix}}_{m \times 1} = \underbrace{\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix}}_{m \times 1} + \underbrace{\begin{bmatrix} Q_{11} & \cdots & Q_{1\,n-m} \\ \vdots & & \\ Q_{m1} & \cdots & Q_{m,n-m} \end{bmatrix}}_{m \times (n-m)} \underbrace{\begin{bmatrix} x_{j_1} \\ x_{j_2} \\ \vdots \\ x_{j_{n-m}} \end{bmatrix}}_{(n-m) \times 1}$$

What do we want to show?

$$T_0 \xrightarrow{\text{pivot.}} T_1 \cdots\cdots T^*$$

terminates.

formalize pivoting step

- Formalize termination condition
- When it terminates it gives the correct answer.

Lemma 1: For each feasible basis B, there exist exactly one tableau $T(B)$, and it is given by:

B_1 $T(B_1)$

B_2

$\vdots$

B_k $T(B)$

$$p = A_B^{-1} b$$

$$Q = -A_B^{-1} A_N$$

$$z_0 = c_B^T A_B^{-1} b$$

$$r^T = c_N^T - (c_B^T A_B^{-1} A_N)$$

Proof: We start with $Ax = b$, $x \geq 0$. We are given that B is a feasible basis.

Rewriting $Ax = b$:
$$A_B x_B + A_N x_N = b$$

$$A_B: m \times m \qquad A_N: m \times (n-m)$$

$$A_B x_B = b - A_N x_N$$

We know A_B has rank m. Hence there is an inverse A_B^{-1}.

$$A_B^{-1} A_B x_B = A_B^{-1} b - A_B^{-1} A_N x_N$$

$$x_B = A_B^{-1} b - A_B^{-1} A_N x_N$$

This gives a tableau with:

$$p = A_B^{-1} b \qquad Q = -A_B^{-1} A_N$$

$$x_B = A_B^{-1} b - A_B^{-1} A_N x_N$$

What about cost c^T?

c_B: the coordinate of c corresponding to basic variables B

c_N: non-basic

$$c^T x = c_B^T x_B + c_N^T x_N$$

$$= c_B^T (A_B^{-1} b - A_B^{-1} A_N x_N) + c_N^T x_N$$

$$= \underbrace{c_B^T A_B^{-1} b}_{z_0} + \underbrace{(c_N^T - c_B^T A_B^{-1} A_N)}_{r^T} x_N$$

This shows that for every B there is a tableau $T(B)$

<u>Uniqueness:</u>

$$\begin{array}{l} x_B = p' + Q' x_N \\ \hline z = z_0' + r'^T x_N \end{array}$$

It is enough to show $p = p'$ and $Q = Q'$.

$$x_B = p + Q x_N \quad \}m$$
$$x_B = p' + Q' x_N \quad \}m$$

Subtracting coordinate wise:

$$(p - p') + (Q - Q') x_N = 0 \quad \text{for all values of } x_N.$$

Now putting $x_N = 0$ gives $p = p'$

To show $Q = Q'$

$$\left(\begin{bmatrix} & & \end{bmatrix} - \begin{bmatrix} & & \end{bmatrix}'\right)\underbrace{\begin{bmatrix} x_N^1 \\ x_N^2 \\ \vdots \\ x_N^{n-m} \end{bmatrix}}_{x_N} = 0$$

$$x_N^1 \left(Q^1 - Q'^1 \right) + x_N^2 \left(Q^2 - Q'^2 \right) + \cdots + x_N^{n-m} \left(Q^{n-m} - Q'^{n-m} \right) = 0$$

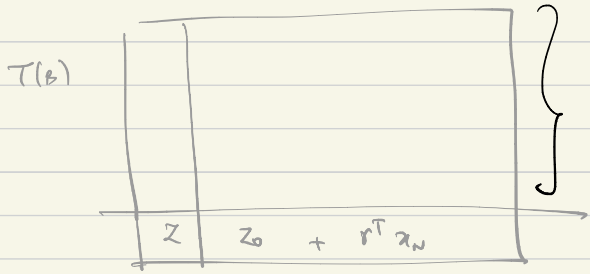
Putting $x_N^1 = 1$ and others 0 gives $Q^1 = Q'^1$

$\vdots$

$x_N^i = 1$ others 0 — $Q^i = Q'^i$

This shows $Q = Q'$.

Lemma 2: If $T(B)$ is a simplex tableau s.t. $r \leq 0$, then the optimum is z_0, attained at the bfs corresponding to B.



Proof: For any feasible soln., the cost is given by
$z = z_0 + r^T x_N$
Since $x_N \geq 0$, and $r \leq 0$, cost $\leq z_0$
Since bfs corresponding to B attains z_0, it is the optimum

To show that the final tableau has the same feasible region.

$$Ax = b$$
$$x \geq 0$$

x_B

$$\begin{array}{cccc} A^1 & A^2 & \cdots & A^n \end{array} \quad \begin{bmatrix} & & & \\ & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B$$

$$A_B x_B + A_N x_N = b$$
$$x \geq 0$$

$$A_B x_B = b - A_N x_N$$

A_B: $m \times m$ matrix, with rank m. It has an inverse

feasible solns. donot change $\longrightarrow$

$$A_B^{-1} A_B x_B = A_B^{-1} b - A_B^{-1} A_N x_N$$
$$x_B = A_B^{-1} b - A_B^{-1} A_N x_N$$

Summary:

- 1. Degenerate pivot steps
- 2. Finding an initial feasible basis
- 3. Started formalizing the simplex tableau