LINEAR PROGRAMMING & COMBINATORIAL OPTIMIZATION

LECTURE 4

Recap:

Given LP in equational form:

$$\text{maximize} \quad c^T x$$
$$\text{subject to} \quad Ax = b$$
$$x \geq 0$$

- If cost c is bounded in the feasible region, the LP attains its optimum at a basic feasible solution.

Questions:

- 1. How to check if cost is bounded in the feasible region?
- 2. If cost is indeed bounded, is there a better way to find the bfs with maximum cost, instead of just arbitrarily enumerating each of them?

Today's lecture:

The Simplex method - an algorithm for solving LPs.

The Simplex Method

An algorithm for solving LPs in equational form.

Idea:
- Start with a bfs
- deduce whether this bfs is optimum or if LP is unbounded.
- If not, move to a bfs with bigger cost.

Example 1:

$$\begin{aligned} \text{Maximize} \quad & x_1 + x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 1 \\ & x_1 \leq 3 \\ & x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

First bring it to equational form by adding slack variables x_3, x_4, x_5

$$\begin{aligned} -x_1 + x_2 + x_3 &= 1 \\ x_1 + x_4 &= 3 \\ x_2 + x_5 &= 2 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

$$A = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Notice that $B = \{3, 4, 5\}$ gives a bfs: $x_3 = 1$, $x_4 = 3$, $x_5 = 2$, $x_1 = x_2 = 0$

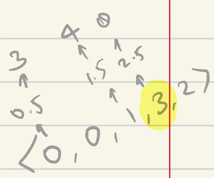
We will start with this bfs. First we will rearrange eqns. conveniently.

$$\text{maximize} \quad x_1 + x_2$$

$$\text{subj. to} \quad \begin{aligned} -x_1 + x_2 + x_3 \quad\quad\quad &= 1 \\ x_1 \quad\quad + x_4 \quad &= 3 \\ x_2 \quad\quad + x_5 &= 2 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Initial bfs has x_3, x_4, x_5 as basic variables and x_1, x_2 as non-basic.



$$\boxed{\begin{aligned} x_3 &= 1 + x_1 - x_2 \\ x_4 &= 3 - x_1 \\ x_5 &= 2 - x_2 \\ \hline z &= x_1 + x_2 \end{aligned}}$$

Rewriting basic variables in terms of non-basic variables.

Writing the objective.

This representation is called a Tableau

The above tableau corresponds to a bfs: $\langle 0, 0, 1, 3, 2 \rangle$
with cost : 0

Next step: Pivoting

If x_1 or x_2 is increased, the cost increases. Suppose we consider x_1.

How far can x_1 be increased?

$x_1 \; x_2 \; x_3 \; x_4 \; x_5$

$[3, 0, 4, 0, 2]$

$[0, 0, 1, 3, 2]$

$$
\begin{aligned}
x_3 &= 1 + x_1 - x_2 \\
x_4 &= 3 - x_1 \\
x_5 &= 2 - x_2 \\
\hline
z &= x_1 + x_2
\end{aligned}
$$

$x_1 = 3 - x_4$

Initial tableau T_0.

Next step: Pivoting

If x_1 or x_2 is increased, the cost increases. Suppose we consider x_1.

How far can x_1 be increased?

- The equation $x_3 = 1 + x_1 - x_2$ gives no restriction.
- $x_4 = 3 - x_1$ says $x_1 \leq 3$. Otherwise, $x_4 < 0$ and we don't get a feasible soln.

Let us therefore increase x_1 to 3.

- By doing this, x_4 becomes 0.
- We now move x_4 to RHS and x_1 to LHS.

$$\boxed{\begin{array}{lllll} x_3 & = 1 & + x_1 & - x_2 \\ x_4 & = 3 & - x_1 & \\ x_5 & = 2 & & - x_2 \\ \hline z & = & x_1 & + x_2 \end{array}}$$

Initial tableau T_0

$\downarrow$

Pivot step:

x_1 enters basis

x_4 leaves basis

$$\begin{array}{lllll} x_3 & = 1 & + (3 - x_4) & - x_2 \\ x_1 & = 3 & - x_4 & \\ x_5 & = 2 & & - x_2 \\ \hline z & = 3 & - x_4 & + x_2 \end{array}$$

$\downarrow$

$$\boxed{\begin{array}{lllll} x_3 & = 4 & - x_4 & - x_2 \\ x_1 & = 3 & - x_4 & \\ x_5 & = 2 & & - x_2 \\ \hline z & = 3 & - x_4 & + x_2 \end{array}} \quad T_1$$

$\langle 3, 0, 4, 0, 2 \rangle$

Cost: 3

Notice that T_1 is an equivalent system of equations to T_0.

By putting $x_4 = x_5 = 0$, we get a feasible soln. $\langle 3, 0, 4, 0, 2 \rangle$
In fact, this is a bfs with basis $\{1, 3, 5\}$. Cost $= 3$, larger than T_0.

$$
\boxed{
\begin{array}{lllll}
x_3 & = & 4 & - x_4 & - x_2 \\
x_1 & = & 3 & - x_4 & \\
x_5 & = & 2 & & - x_2 \\
\hline
z & = & 3 & - x_4 & + x_2
\end{array}
}
$$

T_1

$\langle 3, 2, 2, 0, 0 \rangle$

x_2 upto 2

At T_1, look at z. Notice that increasing x_2 increases cost.

By how much can we increase x_2, by keeping $x_4 = 0$?

$x_3 = 4 - x_4 - x_2$ restricts $x_2 \leq 4$

$x_5 = 2 - x_2$ restricts $x_2 \leq 2$.

So we can go upto 2; as $x_2 \leq 2$ is stricter.

By setting $x_2 = 2$, we get $x_5 = 0$. We will bring x_5 to RHS.

$$
\begin{array}{llllll}
x_3 & = & 4 & - x_4 & - (2 - x_5) \\
x_1 & = & 3 & - x_4 & \\
x_2 & = & 2 & - x_5 & \\
\hline
z & = & 3 & - x_4 & + (2 - x_5)
\end{array}
$$

$\downarrow$

$$
\boxed{
\begin{array}{lllll}
x_3 & = & 2 & - x_4 & + x_5 \\
x_1 & = & 3 & - x_4 & \\
x_2 & = & 2 & & - x_5 \\
\hline
z & = & 5 & - x_4 & - x_5
\end{array}
}
$$

T_2

x_2 enters basis

x_5 leaves basis.

bfs: $\langle 3, 2, 2, 0, 0 \rangle$

$$
\boxed{
\begin{array}{rclll}
x_3 & = & 2 & - x_4 & + x_5 \\
x_1 & = & 3 & - x_4 & \\
x_2 & = & 2 & & - x_5 \\
\hline
z & = & 5 & - x_4 & - x_5
\end{array}
}
\quad T_2
$$

What next?

Once again, notice that T_2 is equivalent to the system that we started with. We have only rewritten the equations so that we have the same variables on RHS.

- T_2 corresponds to a bfs: $\langle 3, 2, 2, 0, 0 \rangle$
  Cost: 5

- Now, notice from $z = 5 - x_4 - x_5$ that increasing x_4 or x_5 will only decrease the cost.

- simplex method terminates here.

Claim: 5 is the optimum cost for this LP.

- For every feasible solution $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5)$ the cost is given by $\tilde{z} = 5 - \tilde{x}_4 - \tilde{x}_5$
  Since $\tilde{x}_4, \tilde{x}_5 \geq 0$, $\tilde{z} \leq 5$.

- T_2 gives a feasible point, which is in fact a bfs, with cost $= 5$.

Today's goals:

- Exercises on simplex method to understand the algorithm.
- In later lectures, we will see a proof of correctness, and why each tableau corresponds to a bfs

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Example 2:

Maximize x_1

Subject to
$$\begin{aligned} x_1 - x_2 &\leq 1 \\ -x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Bring to equational form:

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ -x_1 + x_2 + x_4 &= 2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

$$x_1 - x_2 + x_3 = 1$$
$$-x_1 + x_2 + x_4 = 2$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$\boxed{\begin{aligned} x_3 &= 1 - x_1 + x_2 \\ x_4 &= 2 + x_1 - x_2 \\ \hline z &= x_1 \end{aligned}}$$

T_0

x_1 enters basis

x_3 leaves basis

$$x_1 = 1 - x_3 + x_2$$
$$x_4 = 2 + (1 - x_3 + x_2) - x_2$$

$$\boxed{\begin{aligned} x_1 &= 1 - x_3 + x_2 \\ x_4 &= 3 - x_3 \\ \hline z &= 1 - x_3 + x_2 \end{aligned}}$$

T_1

- x_2 is the only variable that can be pivoted.
- Increasing x_2 gives no constraints on x_1, x_4. Both will stay non-negative.
- Cost can be increased arbitrarily high inside the feasible region.
- Simplex concludes that the LP is unbounded.

$$\boxed{\begin{aligned} x_1 &= 1 - x_3 + x_2 \\ x_4 &= 3 - x_3 \\ \hline z &= 1 - x_3 + x_2 \end{aligned}}$$

T_1

$\langle 1, 0, 0, 3 \rangle$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+\lambda \\ \lambda \\ 0 \\ 3 \end{bmatrix}$$

is a feasible soln. for $\lambda \geq 0$

Cost: $1 + \lambda$

At a tableau T, look at the cost row.

- 1. If all coefficients of variables are negative, then tableau gives optimum. (constant in the cost row)

- 2. Suppose there exists a variable in the cost row with positive coefficient: (say x_i)

  - 2.1. If coefficients of x_i in the constraints are all non-negative, LP is unbounded.

  - 2.2. Else pivot and proceed to next tableau.

Example 3:

Maximize $-x_1 - x_2$

Subject to:

$$x_1 + 2x_2 \leq 4$$
$$x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

$$\boxed{\begin{aligned} x_3 &= 4 - x_1 - 2x_2 \\ x_4 &= 1 - x_2 \\ \hline z &= - x_1 - x_2 \end{aligned}}$$

T_0

Optimum = 0

BFS : $\langle 0, 0, 4, 1 \rangle$

Example 4:

Maximize $-2x_1 + x_2$

Subj. to: $-x_1 + x_2 \leq 2$

$2x_1 + x_2 \leq 6$

$x_1, x_2 \geq 0$

$$\boxed{\begin{array}{lclll} x_3 & = & 2 & + x_1 & - x_2 \\ x_4 & = & 6 & - 2x_1 & - x_2 \\ \hline z & = & & -2x_1 & + x_2 \end{array}}$$

$\downarrow$ x_2 enters basis

x_3 leaves basis

$$\begin{array}{lclll} x_2 & = & 2 & + x_1 & - x_3 \\ x_4 & = & 4 & - 3x_1 & + x_3 \\ \hline z & = & 2 & - x_1 & - x_3 \\ \hline \end{array}$$

$x_4 = 6 - 2x_1 - (2 + x_1 - x_3)$

$z = -2x_1 + (2 + x_1 - x_3)$

Optimum is 2: $\langle 0, 2, 0, 4 \rangle$

Example 5:

Maximize $\quad x_1 + x_2$

Subject to

$$
\begin{aligned}
-x_1 + x_2 &\leq 2 \\
x_2 &\leq 4 \\
x_1 + x_2 &\leq 9 \\
x_1 &\leq 6 \\
x_1 - x_2 &\leq 5 \\
x_1,\ x_2 &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= 2 + x_1 - x_2 \\
x_4 &= 4 - x_2 \\
x_5 &= 9 - x_1 - x_2 \\
x_6 &= 6 - x_1 \\
x_7 &= 5 - x_1 + x_2 \\
\hline
z &= + x_1 + x_2
\end{aligned}
$$

$$\xrightarrow[x_7 \downarrow]{x_1 \uparrow}$$

$$
\begin{aligned}
x_3 &= 7 - x_7 \\
x_4 &= 4 - x_2 \\
x_5 &= 4 + x_7 - 2x_2 \\
x_6 &= 1 + x_7 - x_2 \\
x_1 &= 5 - x_7 + x_2 \\
\hline
z &= 5 - x_7 + 2x_2
\end{aligned}
$$

$x_2 \uparrow \quad x_6 \downarrow$

$$
\begin{aligned}
x_3 &= 7 - x_7 \\
x_4 &= 3 - x_7 + x_6 \\
x_5 &= 2 - x_7 - 2x_6 \\
x_2 &= 1 + x_7 - x_6 \\
x_1 &= 6 - x_6 \\
\hline
z &= 7 + x_7 - 2x_6
\end{aligned}
$$

$$\xleftarrow[x_5 \downarrow]{x_7 \uparrow}$$

$$
\begin{aligned}
x_3 &= 5 + x_5 + 2x_6 \\
x_4 &= 1 + x_5 + 3x_6 \\
x_7 &= 2 - x_5 - 2x_6 \\
x_2 &= 3 - x_5 - 3x_6 \\
x_1 &= 6 - x_6 \\
\hline
z &= 9 - x_5 - 3x_6
\end{aligned}
$$

optimum. cost = 9

bfs : $\langle 6, 3, 5, 1, 0, 0, 2 \rangle$

Summary:

- simplex method
- Detecting optimum, unboundedness } examples