LINEAR PROGRAMMING & COMBINATORIAL OPTIMIZATION

LECTURE 3

Recap:

Given: LP in equational form

- A has rank 'm'
- $Ax = b$ has at least one soln.

$$\boxed{\text{maximize } c^T x \quad \text{subject to} \quad Ax = b, \; x \geq 0}$$

- 1. Want to characterize vertices of the feasible region $Ax = b, x \geq 0$.
- 2. Interior point: a feasible point $v \in \mathbb{R}^n$ s.t. $\exists\, u \in \mathbb{R}^n$ with:

  $v + u$ and $v - u$ both being feasible

- 3. This implies: $Au = 0$ (from $Ax = b$)

  $\forall i \in \{1, \dots, n\}: \quad v_i = 0 \implies u_i = 0$ (from $x \geq 0$)

- 4. As: $\sum_{i=1}^{n} u_i A^i = 0$

  We have $\sum_{i \text{ s.t. } u_i \neq 0} u_i A^i = 0$

$u \; [\neq 0 \;\; \neq 0 \;\; 0 \;\; 0 \;\; \neq 0]$ $\therefore$ Columns A^i s.t. $u_i \neq 0$ are linearly dependent.

$v \; [\neq 0 \;\; \neq 0 \;\; 0 \;\; \neq 0 \;\; \neq 0]$ $\implies$ ~~Columns A^i s.t. $v_i \neq 0$ are linearly dependent.~~

- 5. Vertex: v is a vertex if such a 'u' does not exist.

  $\therefore$ Columns A^i s.t. $v_i \neq 0$ are linearly independent.

This leads to the definition of Basic Feasible Solutions of an LP in equational form.

- v is a feasible solution.

Suppose $\exists u$ s.t. $v+u$, $v-u$ are feasible

$u \neq 0$

Then the columns indexed by $K = \{ i \mid v_i \neq 0 \}$ are linearly dependent.

$$Av = b$$

$$\sum_{i=1}^{n} v_i A^i = b \qquad \therefore \sum_{\substack{i=1 \\ \text{s.t. } v_i \neq 0}}^{n} v_i A^i = b$$

$$\sum_{i=1}^{n} (v_i + u_i) A^i = b$$

$$\sum_{\substack{i=1 \\ \text{s.t. } v_i \neq 0}}^{n} (v_i + u_i) A^i$$

$$+ \sum_{\substack{i=1 \\ v_i = 0}}^{n} (0 + u_i) A^i$$

But $u_i = 0$

$$\Rightarrow \sum_{\substack{i=1 \\ \text{s.t. } v_i \neq 0}}^{n} (v_i + u_i) A^i = b$$

$$\Rightarrow \sum_{\substack{i=1 \\ \text{s.t. } v_i \neq 0}}^{n} v_i A^i + \sum_{\substack{i=1 \\ v_i \neq 0}}^{n} u_i A^i = b$$

b

$$\Rightarrow \sum_{\substack{i=1 \\ v_i \neq 0}}^{n} u_i A^i = 0$$

BASIC FEASIBLE SOLUTIONS (BFS):

Definition 1: A feasible soln. $v \in \mathbb{R}^n$ is a 'bfs' if the columns indexed by

$K = \{ i \mid v_i > 0 \}$ are linearly independent.

Definition 2: A bfs is a feasible solution $v \in \mathbb{R}^n$ s.t. there exists an 'm' element set $B \subseteq \{1, 2, \dots, n\}$ s.t.

- columns indexed by B are linearly independent.
- $v_j = 0 \quad \forall j \notin B$

Lemma: Both definitions are equivalent: v satisfies definition 1 iff v satisfies definition 2.

Proof: v is a feasible Soln.

v satisfies definition 2 $\Rightarrow$ v satisfies definition 1.

Suppose v satisfies definition 2.

Then, there exists a $B \subseteq \{1, 2, \dots, n\}$

$B = \{ i_1, i_2, \dots, i_m \}$

s.t. $v_j = 0 \quad \forall j \notin B$

$A^{i_1} \; A^{i_2} \dots A^{i_m}$ are linearly independent.

$K \subseteq B$

$\therefore$ Columns indexed by K will be linearly independent

$\Rightarrow v$ satisfies Definition 1.

v satisfies Definition 1 $\Rightarrow$ v satisfies Definition 2.

Suppose v satisfies Definition 1.

$K = \{ i \mid v_i > 0 \}$.

Columns indexed by K are linearly independent.

- Suppose $|K| = m$

  Then take $B = K$

- Note that $|K|$ cannot be $> m$ since col-rank $(A) = m$

- Suppose $|K| < m$

  $B = K \cup$ some more column

  Column space has dimension m.

  $\therefore$ There are $A^{j_1} \quad A^{j_2} \ldots \quad A^{j_m}$ [basis]

  $K = \{ i_1 \quad i_2 \ldots \quad i_k \}$

  $A^{i_1} \quad A^{i_2} \ldots \quad A^{i_k}$ (linearly independent)

Using the "basis", it is possible to extend $\{ A^{i_1} \ldots A^{i_k} \}$ to an m-element set of linearly independent vectors.

[Exercise]

This extension gives the set B.

Notice that $v_j = 0 \quad \forall j \notin B$.

EXAMPLE: (Problem sheet 1, Question 3)

Write the basic feasible solutions:

$$\begin{aligned} x_1 + x_2 + x_3 \qquad &= 6 \\ x_2 \qquad + x_4 &= 3 \end{aligned}$$

$$x_1, \quad x_2, \quad x_3, \quad x_4 \quad \geq 0$$

Definition 2: A bfs is a feasible solution $v \in \mathbb{R}^n$ s.t. there exists an 'm' element set $B \subseteq \{1, 2, \ldots, n\}$ s.t.

- columns indexed by B are linearly independent.
- $v_j = 0 \quad \forall j \notin B$

$$A = \begin{array}{cccc} A^1 & A^2 & A^3 & A^4 \end{array}$$
$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$m = 2$

$$\underbrace{\begin{array}{cc} A^1 & A^2 \\ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array}}_{B = \{1,2\}} \qquad x = [x_1 \quad x_2 \quad 0 \quad 0]$$

$$\begin{aligned} x_1 + x_2 &= 6 \\ x_2 &= 3 \end{aligned}$$

$[3, \quad 3 \quad 0 \quad 0]$

$$\begin{array}{cc} A^1 & A^4 \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \qquad x = [x_1 \quad 0 \quad 0 \quad x_4]$$

$$\begin{aligned} x_1 &= 6 \\ x_4 &= 3 \end{aligned}$$

$[6 \quad 0 \quad 0 \quad 3]$

$$A = \begin{array}{cccc} A^1 & A^2 & A^3 & A^4 \\ \left[\begin{matrix} 1 \\ 0 \end{matrix}\right. & \begin{matrix} 1 \\ 1 \end{matrix} & \begin{matrix} 1 \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 1 \end{matrix}\right] \end{array}$$

$$\begin{array}{cc} A^2 & A^3 \\ \left[\begin{matrix} 1 \\ 1 \end{matrix}\right. & \left.\begin{matrix} 1 \\ 0 \end{matrix}\right] \end{array} \qquad X = [0 \quad x_2 \quad x_3 \quad 0]$$

$$x_2 + x_3 = 6$$

$$x_2 = 3$$

$$[0 \quad 3, \quad 3 \quad 0]$$

$$\begin{array}{cc} A^2 & A^4 \\ \left[\begin{matrix} 1 \\ 1 \end{matrix}\right. & \left.\begin{matrix} 0 \\ 1 \end{matrix}\right] \end{array} \qquad X = [0 \quad x_2 \quad 0 \quad x_4]$$

$$x_2 = 6$$

$$x_2 + x_4 = 3$$

$$[0 \quad 6 \quad 0 \quad -3]$$

not feasible

$$\begin{array}{cc} A^3 & A^4 \\ \left[\begin{matrix} 1 \\ 0 \end{matrix}\right. & \left.\begin{matrix} 0 \\ 1 \end{matrix}\right] \end{array} \qquad X = [0 \quad 0 \quad x_3 \quad x_4]$$

$$[0 \quad 0 \quad 6 \quad 3]$$

OPTIMUM OCCURS AT A BFS:

Theorem: Suppose there exists $D \in \mathbb{R}$ s.t. $c^T v \leq D$ for every feasible point v.

c is bounded.

Then:

- 1. the LP has an optimum.
- 2. optimum occurs at a bfs.

Lemma: Assume that cost is bounded.

For every feasible solution v, there exist a bfs v^* s.t.

$$c^T v^* \geq c^T v.$$

Claim: This lemma proves the above theorem.

There are finitely many bfs's.

The lemma implies that it is sufficient to look at bfs's to get the maximum cost.

- Since there are only finitely many bfs's, optimum occurs at one of the points.

Lemma: Assume that cost is bounded.

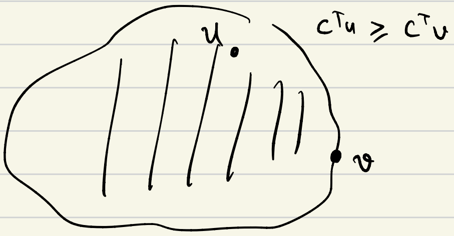
For every feasible solution v, there exists a bfs v^* s.t.

$$c^T v^* \geq c^T v.$$

Proof: Pick some arbitrary feasible soln. v.

$$U = \{ u \in \mathbb{R}^n \mid u \text{ is feasible and } c^T u \geq c^T v \}$$

U is non-empty since $v \in U$.



Now, let $v^* \in U$ with the maximum no. of '0' coordinates.

Claim: v^* is a bfs.

Suppose not. Let $K = \{ i \mid v^*(i) > 0 \}$

$$= \{ i_1, i_2, \ldots, i_k \}$$

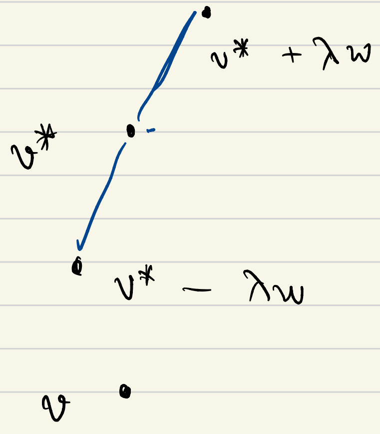
Columns $A^{i_1}, A^{i_2}, \ldots, A^{i_k}$ are linearly dependent.

This gives us a w and a λ s.t.

1) $\left. \begin{array}{l} v^* + \epsilon w \\ v^* - \epsilon w \end{array} \right\}$ both are feasible. for some ϵ.

2) $Aw = 0$

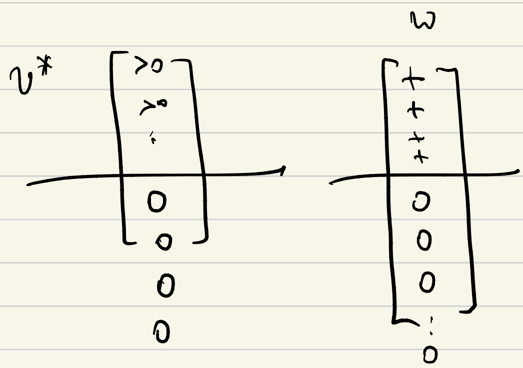
$\left.\begin{array}{l} v^* + \lambda w \\ v^* - \lambda w \end{array}\right\}$ both are feasible.



i) Suppose $c^T w > 0$

$c^T(v^* + \lambda w)$
$= c^T v^* + \lambda c^T w$

$\Rightarrow \quad c^T(v^* + \lambda w) > c^T v^*$



If $w_i \geq 0 \quad \forall i$.

Then $v^* + \lambda w \geq 0 \quad \forall \lambda$

$A(v^* + \lambda w) = b \quad \forall \lambda$

Increasing λ, we get feasible points with arbitrarily big cost.

This is a contradiction as cost is bounded.

$c^T(v^* + \lambda w)$
$= c^T v^* + \lambda c^T w$

$\therefore$ There exists some i s.t. $w_i < 0$.

This gives a bound on the λ.

$\therefore$ There exists some $\bar{\lambda}$ s.t.

- $v^* + \bar{\lambda} w$ is feasible.
- $v^* + \bar{\lambda} w$ has more 0's than v^*
- $c^T (v^* + \bar{\lambda} w) > c^T v^*$

This is a contradiction to our assumption on v^*:

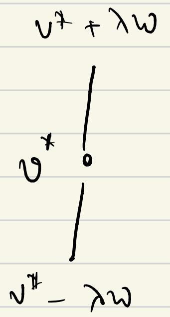
$$U = \{ u \in \mathbb{R}^n \mid u \text{ is feasible and } c^T u \geq c^T v \}$$

We picked v^* to be an element in U with the max. no. of zeroes.

However, $v^* + \bar{\lambda} w \in U$, and has more zeroes.

-2. Suppose $c^T w < 0$

Now consider $w' = -w$ and proceed as above.



$Aw' = 0$

$\exists \epsilon$ s.t. $v^* + \epsilon w' \ngeq 0$

-3. Suppose $c^T w = 0$

$v^* \;—\; w.$

v

$c^T w = 0 \qquad\qquad Aw = 0$

- Suppose there exists some coordinate i s.t. $w_i < 0$.

  Find a good $\bar{\lambda}$ s.t.

  - $v^* + \bar{\lambda} w \geq 0$
  - $v^* + \bar{\lambda} w$ has more zeros.
  - $c^T(v^* + \bar{\lambda} w) = c^T v$

  $\rightarrow$ Contradiction.

- Suppose all coordinates of w are +ive.

  Then consider $w' = -w$ and proceed with the same argument.

Illustration of the previous proof:

Pick a feasible v

$U = \{ u \mid u \text{ is feasible and } c^T u \geq c^T v \}$

v^* : element in U with max no. of 0's

Claim : v^* is BFS

Suppose not.

$K = \{ i \mid v^*_i > 0 \}$

There is w s.t. $Aw = 0$, and $v^*_i = 0 \Rightarrow w_i = 0$

Suppose $c^T w > 0$

$\forall i : w_i \geq 0$

Cost is unbdd.

$\exists i : w_i < 0$

There is another $v' \in U$ with more zeroes

$c^T w < 0$

Take $w' = -w$

$c^T w = 0$

$\forall i : w_i \geq 0$

Cost is unbdd.

$\exists i : w_i < 0$

There is another $v' \in U$ with more 0's.

Each branch leads to a contradiction

$\therefore$ Supposition that v^* is not a BFS is false.

Summary:

- definitions of bfs

- When cost is bounded, optimum occurs at a bfs.