Lecture 2: Basic Feasible Solutions

Today's lecture:

- Understanding some special solutions to the equational form.

Recall: LP in equational form

$$\text{maximize} \quad c^T x \qquad \longrightarrow \text{Objective / cost function}$$

$$\text{subject to} \quad \left.\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}\right\} \longrightarrow \text{constraints}$$

$x \in \mathbb{R}^n$ A: $m \times n$ real matrix

$b \in \mathbb{R}^m$

$c \in \mathbb{R}^n$

Note: When we talk about algorithms & complexity, we will assume all numbers are integers.

Reference: Chapter 3 of book:

Understanding and using Linear Programming
- Matoušek & Gärtner

Terminology:

Feasible point: a vector $v \in \mathbb{R}^n$ that satisfies

$$Av = b, \quad v \geq 0$$

$\rightarrow$ a solution to the constraints.

Feasible region: set of feasible points.

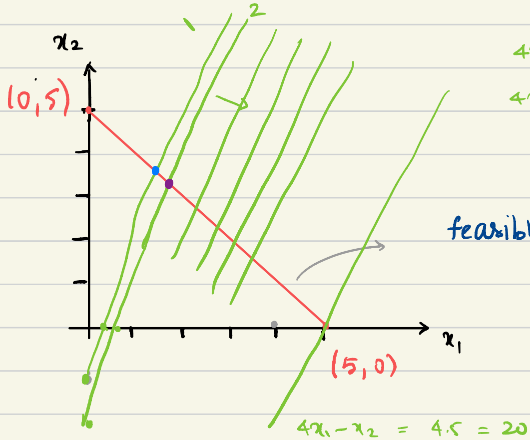
Optimum: the feasible point that gives the maximum cost.

Examples:

1. maximize $4x_1 - x_2$

subject to $x_1 + x_2 = 5$

$x_1, x_2 \geq 0$

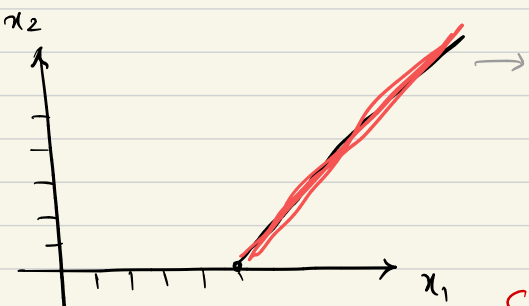


Optimum : (5,0)

Optimal cost: 20

2. maximize x_1

subject to $x_1 - x_2 = 5$



Feasible region is unbounded.

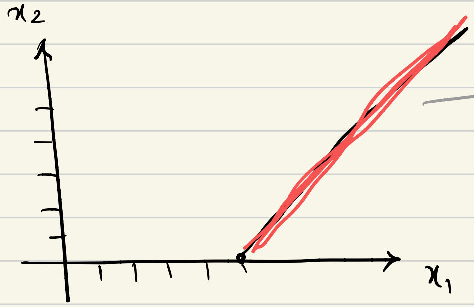
LP is unbounded

$\left\{ \text{Cost is unbounded in the feasible region} \right.$

maximize $x_1 - x_2$

subject to $x_1 - x_2 = 5$

$x_1, x_2 \geq 0$



feasible region is unbounded,

→ Cost is bounded in the feasible region.

Optimum: infinitely many.

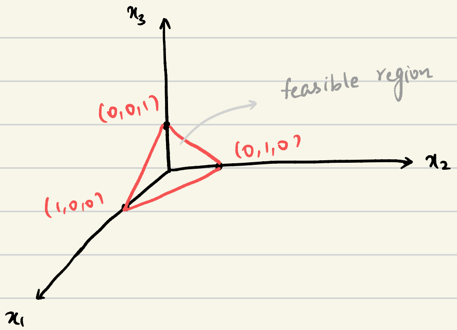
- any point in the feasible region is an optimum.

Optimal cost: 5

3. maximize $x_1 - 5x_2 + 4x_3$

subject to $x_1 + x_2 + x_3 = 1$

$x_1, x_2, x_3 \geq 0$



Optimum occurs at $(0,0,1)$

→ Not easy to illustrate

Next goals:

- 1. Give a characterization of feasible points that are "vertices" of the feasible region. (Today)

- 2. Show that if the LP has an optimum, then one of the "vertices" is an optimum. (Next lecture)

Outline of the rest of the lecture:

Step 1: Linear algebra basics

Step 2: Assumptions about LP

Step 3: Informal intuition about vertices and interior points

Step 4: Formalizing this intuition.

Step 1: Linear Algebra basics:

We will stick to real numbers.

Vector: an ordered tuple $(v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ of real numbers

$$\bar{v} = (v_1, v_2, \ldots, v_n)$$

$$\bar{u} = (u_1, u_2, \ldots, u_n)$$

$$\bar{v} + \bar{u} = (v_1+u_1, v_2+u_2, \ldots, v_n+u_n)$$

$$t\bar{v} = (tv_1, tv_2, \ldots, tv_n) \quad \text{where } t \in \mathbb{R} \text{ (a scalar)}$$

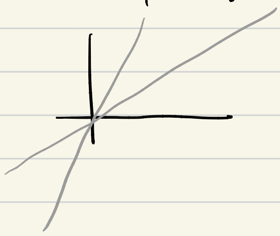
Linear subspace of $\mathbb{R}^n$: A set of vectors $V \subseteq \mathbb{R}^n$ s.t.

for all $\bar{u}, \bar{v} \in V$ and $t \in \mathbb{R}$, $\bar{u} + \bar{v} \in V$

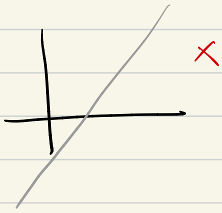
$t\bar{v} \in V$

Remark: A subspace contains the $\bar{0}$ vector.

Examples: Subspaces of $\mathbb{R}^2$ are $\{\bar{0}\}$, lines passing through origin, $\mathbb{R}^2$ itself



- Line not passing through origin is not a subspace.



- Subspaces of $\mathbb{R}^3$ are $\{\bar{0}\}$, lines passing through origin, planes passing through origin, $\mathbb{R}^3$

Linear dependence:

Vectors $\bar{u}_1, \bar{u}_2, \dots, \bar{u}_k$ are said to be linearly dependent if $\exists$ reals $\alpha_1, \alpha_2, \dots, \alpha_k$, not all of them 0

s.t. $\boxed{\alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2 + \dots + \alpha_k \bar{u}_k = 0}$

Examples: 1) $\bar{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\bar{u}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$$2\bar{u}_1 - \bar{u}_2 = 0$$

2) $\bar{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\bar{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2 = 0$$

$$\alpha_1 + 2\alpha_2 = 0 \quad \text{— (1)}$$
$$2\alpha_1 + \alpha_2 = 0 \quad \text{— (2)}$$

From (1), $\alpha_1 = -2\alpha_2$

Substituting in (2): gives: $\alpha_2 = 0$

$$\Rightarrow \alpha_1 = 0$$

$\therefore$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are not linearly dependent.

Linear Independence:

$\bar{u}_1, \bar{u}_2, \dots, \bar{u}_k$ are linearly independent if for all reals $\alpha_1, \alpha_2, \dots, \alpha_k$

$$\boxed{\alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2 + \dots + \alpha_k \bar{u}_k = 0 \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_k = 0}$$

Basis of a linear subspace:

A set of vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$ forms a basis of a subspace V if

- 1. they are linearly independent.
- 2. every vector $\bar{w} \in V$ can be written as a linear combination of $\bar{v}_1, \ldots, \bar{v}_k$

ie., $\bar{w} = \alpha_1 \bar{v}_1 + \alpha_2 \bar{v}_2 + \cdots + \alpha_k \bar{v}_k$

Theorem: Let $\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m\}$ and $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$ be two bases of a subspace. Then $m = n$.

Dimension of a subspace: cardinality of its basis

Rank of a matrix:

$$A = \begin{array}{c} m \\ \text{rows} \end{array} \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \quad m \times n$$

Rows: $A_1 = [a_{11} \ \cdots \ a_{1n}]^T \in \mathbb{R}^n$

$\vdots$

$A_m = [a_{m1} \ \cdots \ a_{mn}]^T \in \mathbb{R}^n$

Row space: $\left\{ \sum_{i=1}^{m} \alpha_i A_i \mid \alpha_i \in \mathbb{R} \ \forall i \in \{1, \ldots, m\} \right\}$

This is a subspace

Row rank: dimension of row-space

= Maximum No. of linearly independent rows among $A_1, A_2, \ldots, A_m$

Columns: $A^{(1)} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \in \mathbb{R}^m$ $\cdots$ $A^{(n)} = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \in \mathbb{R}^m$

Column space: $\left\{ \sum_{i=1}^{n} \alpha_i A^i \mid \alpha_i \in \mathbb{R} \ \forall i \in \{1, \ldots, n\} \right\}$

Column rank: dimension of column space

= Max. no. of linearly independent columns.

Theorem: For every matrix A: $\underbrace{\text{row-rank}(A) = \text{column rank}(A)}_{\text{called rank of matrix } A}$

Step 2: Assumptions on the equational form

$$\begin{aligned} 2x_1 + x_2 + 5x_3 &= 10 \\ -x_1 + 3x_2 &= 5 \\ x_1 + 4x_2 + 5x_3 &= 15 \end{aligned}$$

$$x_1, x_2, x_3 \geq 0$$

$R_3 = R_1 + R_2$

$$\begin{aligned} 2x_1 + x_2 + 5x_3 &= 10 \\ -x_1 + 3x_2 &= 5 \end{aligned}$$

$$x_1, x_2, x_3 \geq 0$$

removed R_3

- Both the systems have the feasible region.

$$\begin{aligned} 2x_1 + x_2 + 5x_3 &= 10 \\ -x_1 + 3x_2 &= 5 \\ x_1 + 4x_2 + 5x_3 &= 12 \end{aligned}$$

$$x_1, x_2, x_3 \geq 0$$

No solution to $Ax = b$

Note: Whether $Ax = b$ has a solution can be found efficiently using Gaussian elimination.

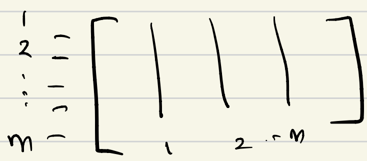
Therefore, we can assume that $Ax = b$ has a solution.

- Hence, we can assume that all rows of A are linearly independent

A: $m \times n$

- all the m rows are lin. independent

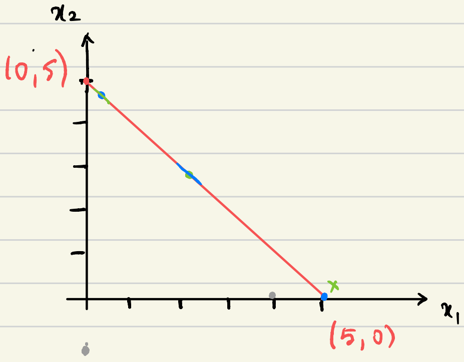
$\Rightarrow$ row-rank $(A) = m$

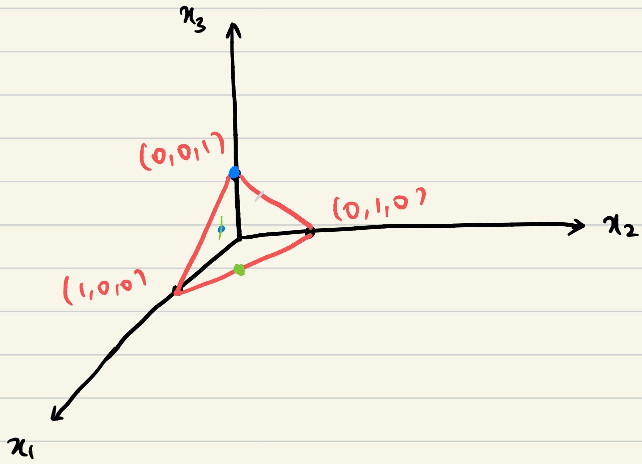


$\Rightarrow$ col-rank $(A) = m$

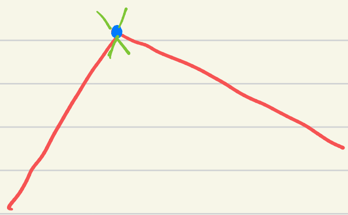
$\Rightarrow n \geq m$

Step 3: An informal intuition about vertices of the feasible region.



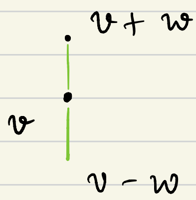


In the interior points, we can draw a line "around" the point that stays entirely in the feasible region.



- We cannot do this at vertices.

interior of
$Ax = b$
$x \geq 0$



w: $[-0.1 \;\; -0.1 \;\; +0.2]$

$[0.4, 0.2, 0.4]$ — $v + w$

$[0.5, 0.3, 0.2]$ — v

$[0.6, 0.4, 0]$ — $v - w$

$[+0.1, +0.1, -0.2]$

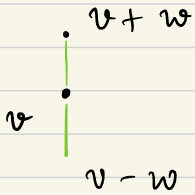
-1. We want a w s.t. $Aw = 0$

$\because$ $A(v+w) = b$ $\quad$ $A(v-w) = b$

$Av + Aw = b$, with $Av = b$

$\Rightarrow Aw = 0$

interior of
$Ax = b$
$x \geq 0$



w: $[-0.1 \quad -0.1 \quad +0.2]$

$[0.4, 0.2, 0.4]$ — $v + w$

$[0.5, 0.3, 0.2]$ — v

$[0.6, 0.4, 0]$ — $v - w$

$[+0.1, +0.1, -0.2]$

1. We want a w s.t. $Aw = 0$

$\because \quad A(v+w) = b \qquad A(v-w) = b$

$$\underbrace{Av}_{= b} + Aw = b$$

$\Rightarrow Aw = 0$

2. We want $v + w \geq 0$

$v - w \geq 0$

So, we don't want w to disturb the 0 coordinates of v.

w: $[+0.1, -0.1, 0]$

v+w: $[0.7, 0.3, 0]$

v: $[0.6, 0.4, 0]$ $\qquad v_i = 0 \Rightarrow w_i = 0$

v − w: $[0.5, 0.5, 0]$

To draw a line around $v \in \mathbb{R}^n$ we look for a 'w' $\in \mathbb{R}^n$ s.t.

- 1. $Aw = 0$

- 2. $v_i = 0 \implies w_i = 0$

i^{th} coordinate of v

If we have such a vector,

$$v \quad [\underline{>0}, \underline{>0}, \underline{>0}, \underline{>0}, 0, 0, 0]$$

w :

We can scale w so that $v + \lambda w \geq 0$
$v - \lambda w \geq 0$

$$A(v + \lambda w) = b$$

$$Ax = b$$
$$x \geq 0$$

Step 4: Formalizing the intuition: Basic Feasible Solutions (BFS)

$$v \quad \text{s.t.} \quad Av = b, \quad v \geq 0$$ a feasible point

$$A \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$v_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} \ \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$v_1 A^1 + v_2 A^2 + \cdots \; v_n A^n = b$$

Consider non-zero coordinates in v. So all of them are strictly > 0

$$v_{i_1} A^{i_1} + \cdots \; v_{i_k} A^{i_k} = b$$

- Suppose $A^{i_1} \ldots A^{i_k}$ are linearly dependent.

$$\alpha_{i_1} A^{i_1} \ldots \alpha_{i_k} A^{i_k} = 0$$

$(v_{i_1} + \alpha_{i_1}) A^{i_1} + \ldots (v_{i_k} + \alpha_{i_k}) A^{i_k} = b$

w: $w_i = \alpha_i$ if $i \in \{ i_1, i_2, \ldots i_k \}$

0

Observe: $Aw = 0$, $v_i = 0 \Rightarrow w_i = 0$. We can find a λ s.t.

$v + \lambda w \geq 0$

$v - \lambda w \geq 0$

- What happens when $A_{i_1}, \dots, A_{i_k}$ are linearly independent.

v: $i_1, i_2, \dots, i_k$ are the non-zero coordinates of v.

Suppose $A_{i_1}, \dots, A_{i_k}$ are linearly ind.

$$\alpha_1 A_{i_1} + \dots + \alpha_{i_k} A_{i_k} = 0 \Rightarrow \alpha_i = 0$$

We cannot find a w s.t. $Aw = 0$

$v_i = 0 \Rightarrow w_i = 0.$

$$v = [\, i_1 \dots i_k \,\vdots\, 0\ 0\ 0\ 0 \,]$$

$$w = [\, w_{i_1} \dots w_{i_k} \,|\, \underline{0\ 0\ 0\ 0} \,]$$

$Aw = 0$

$$w_{i_1} A^{i_1} + \dots + w_{i_k} A^{i_k} = 0 \Rightarrow w_{i_1} = w_{i_2} = \dots = w_{i_k} = 0$$

since $A^{i_1} \dots A^{i_k}$ are linearly independent.

$\therefore$ We cannot move around a line at such points.

BASIC FEASIBLE SOLUTIONS:

A basic feasible solution of the LP

maximize $c^T x$ subject to $Ax = b$, $x \geq 0$

is a feasible solution $x \in \mathbb{R}^n$ s.t.

there exists an 'm' element set $B \subseteq \{1, 2, \ldots, n\}$ s.t.

- 1. the columns indexed by B are linearly independent.
- 2. $x_j = 0 \quad \forall j \notin B$

$$x : \begin{array}{cccc} 0 & i_1 & 0 & i_m \quad 0 \\ \end{array}$$
$$\underbrace{[\, \cancel{x_1} \quad x_2 \quad / \; \cdots \quad \cancel{x_n} \,]}$$

$$A : \begin{array}{cccc} i_1 & i_2 & & i_m \\ \end{array}$$
$$\begin{bmatrix} | & | & \cdots & | \end{bmatrix}$$

Example:

$$\begin{aligned} 2x_1 + x_2 + 4x_3 \quad &= 8 \\ x_2 \qquad\qquad + x_4 &= 4 \end{aligned}$$

$$x_1, x_2, \ldots, x_3, x_4 \geq 0$$

$$\begin{bmatrix} 2 & 1 & 4 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$m = 2$

$$[\, x_1 \quad x_2 \quad 0 \quad 0 \,]$$

$$2x_1 + x_2 + 4x_3 = 8$$
$$x_2 + x_4 = 4$$

$$x_1,\ x_2,\ \ldots x_3,\ x_4 \geqslant 0$$

$$\begin{bmatrix} 2 & 1 & 4 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$
$$\times \quad \times$$

$m=2$

$$\begin{bmatrix} x_1 & x_2 & 0 & 0 \end{bmatrix}$$

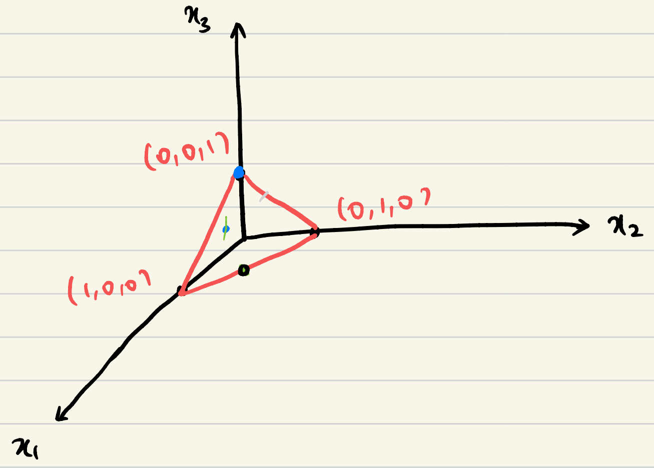
$$\begin{bmatrix} 2 & 1 & 4 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$2x_1 + x_2 = 8$$
$$x_2 = 4$$

$$x_1 = 2$$

$[2 \quad 4 \quad 0 \quad 0]$ is a bfs.

Exercise:



$A = [1 \quad 1 \quad 1]$

$[0.6, \ 0.4, \ 0]$

is not a bfs.

(check with the defn.)

Consider the LP with constraints

$$x_1 + x_2 + x_3 = 1$$
$$x_1, \ x_2, \ x_3 \geq 0$$

- Verify that $[0.6, \ 0.4, \ 0]$ is not a bfs.

Summary:

- Notion of vertices in an equational form
  ↳ informal definition : cannot move around in a line still staying within feasible region
- "Moving around" v means : $\exists w$ s.t. $Aw = 0$
  and $v_i = 0 \implies w_i = 0$

  $\therefore$ $v + \lambda w \geq 0$ for some λ
  $v - \lambda w \geq 0$

  and $v + \lambda w$, $v - \lambda w$ are feasible
- Definition of basic feasible solutions.