LINEAR PROGRAMMING & COMBINATORIAL OPTIMIZATION

LECTURE 14

Last lecture:

- Look at combinatorial problems as ILPs.
- LP relaxations
- When do ILP optima & LP optima coincide?
  - Totally Unimodular Matrices.
- Examples: maximum matching & minimum vertex cover in bipartite graphs.

Today:

- König's theorem
- Non-bipartite case: vertex cover

KÖNIG'S THEOREM: Let $G = (X \dot\cup Y, E)$ be a bipartite graph. The size of a maximum matching equals the size of a minimum vertex cover.

ILPs:

Maximum matching

$$\text{maximize} \sum_{e \in E} x_e$$

$$\text{s.t.} \quad \sum_{\substack{e \text{ incident} \\ \text{on } v}} x_e \le 1 \quad \text{for all } v \in X \dot\cup Y$$

$$x_e \in \{0,1\}$$

Min vertex cover

$$\text{Minimize} \sum_{v \in X \dot\cup Y} x_v$$

$$\text{s.t.} \quad x_u + x_v \ge 1 \quad \forall (u,v) \in E$$

$$x_v \in \{0,1\}$$

LP relaxations:

$$0 \le x_e \le 1$$

$$0 \le x_v \le 1$$

Even if we remove $x_e \le 1$, the feasible solutions will have $x_e \le 1$ because of the constraints.

$$\text{maximize} \sum_{e \in E} x_e$$

$$\text{s.t.} \quad \sum_{\substack{e \text{ incident} \\ \text{on } v}} x_e \le 1 \quad \text{for all } v \in X \dot\cup Y$$

$$x_e \ge 0 \quad \forall e$$

Suppose we remove the constraints $x_v \le 1$

Consider a feasible soln.

$$y = \langle y_{v_1}, y_{v_2}, \ldots, y_{v_z} \rangle$$

From y, construct $\bar{y}$

$$\bar{y} := \quad \bar{y}_{v_i} = 1 \quad \text{if } y_{v_i} \ge 1$$
$$\bar{y}_{v_i} = y_{v_i}$$

- $\bar{y}$ is feasible, and has smaller cost.

$\therefore$ optimum will be at a point where $x_v \le 1 \ \forall v$

LP relaxations:

Maximum matching

maximize $\sum_{e \in E} x_e$

s.t. $\sum_{e \text{ incident on } v} x_e \leq 1$ for all $v \in X \cup Y$

$x_e \geq 0 \quad \forall e$

Min vertex cover

Minimize $\sum_{v \in X \cup Y} x_v$

s.t. $x_u + x_v \geq 1 \quad \forall (u,v) \in E$

$x_v \geq 0 \quad \forall v$

duals of each other.

$$
\begin{array}{c|cccc}
 & e_1 & e_2 & \cdots & e_l \\
\hline
v_1 & 1 & 1 & 1 & \leq 1 \\
v_2 & & 1 & 1 & \\
v_3 & & & & \\
\vdots & & \cdots & & \\
v_m & 1 & & & \\
v_k & & & &
\end{array}
$$

Proof of König's theorem:

1. Both LPs are feasible. Therefore, both primal & dual have optima, and the optima coincide.

2. We have seen last lecture that the matrix corresponding to constraints in max. matching LP is T.U.M.
   $\therefore$ Its transpose (which occurs in the dual) is also T.U.M.

3. Max. matching: ILP optimum = LP optimum
   Min vertex cover ILP optimum = LP optimum

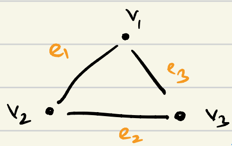
   ILP optimum = ILP optimum; LP optimum = LP optimum (duality)

NON-BIPARTITE GRAPHS:

Max matching | Min vertex cover

↳ NP-complete

1. König's property does not hold anymore.



Max matching $= 1$ | Min vertex cover $= 2$

2. The incidence matrix is not TUM

$$
\begin{array}{c|cccc}
 & e_1 & e_2 & \cdots & e_k \\
\hline
v_1 & 1 & 0 & 1 & \cdot \\
v_2 & 0 & 1 & 0 & \cdot \\
\vdots & 1 & & \vdots & \\
v_k & & & &
\end{array}
$$

$$
\begin{array}{c|ccc}
 & e_1 & e_2 & e_3 \\
\hline
v_1 & 1 & 0 & 1 \\
v_2 & 1 & 1 & 0 \\
v_3 & 0 & 1 & 1
\end{array}
$$

$\downarrow$

$\det(\quad) = 2$

not TUM.

3. ILP optimum vs LP optimum (max matching)

ILP optimum $= 1$ LP optimum $= 3/2$

Minimum Vertex Cover – General graph:

$$G = (V, E)$$

ILP:

$$\text{minimize} \sum_{v \in V} x_v$$

$$\text{s.t.} \quad x_u + x_v \geq 1 \quad \forall (u,v) \in E$$

$$x_u \in \{0,1\} \quad \forall u$$

LP relaxation:

$$\text{minimize} \sum_{v \in V} x_v$$

$$\text{s.t.} \quad x_u + x_v \geq 1 \quad \forall (u,v) \in E$$

$$x_u \geq 0 \quad \forall u$$

$$x \in \mathbb{R}^n$$

Rounding: Suppose x^* is the LP optimum.

$$\langle 0, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{4}{5} \ldots \rangle$$

As seen earlier, all values are ≤ 1 in the optimum.

$$S_{LP} = \{ v \in V \mid x^*_v \geq \tfrac{1}{2} \}$$

Claim: S_{LP} is a vertex cover.

For every edge (u, v) we have the constraint

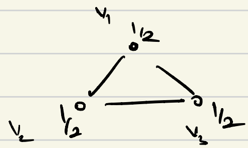
$$x_u + x_v \geq 1$$

$\therefore$ at least one of x_v^* or x_u^* should have $\geq 1/2$

In S_{LP}, we have picked vertices with $x^*_v \geq 1/2$

$\therefore$ at least one of the end points for each edge should be in S_{LP}.

$\Rightarrow$ S_{LP} is a vertex cover.



$\rightarrow S_{LP} = \{v_1, v_2, v_3\}$

Claim: $|S_{LP}| \leq 2 \cdot |\text{min. vertex cover}|$

$\downarrow$

$|S_{ILP}|$

x^* is LP optimum. Suppose y^* is ILP optimum

$$\sum_{v \in V} x_v^* \leq \sum_{v \in V} y_v^* = |S_{ILP}| \longrightarrow ①$$

$|S_{LP}| = \langle 0, 1, 1, 0, 1 \ldots 0 \rangle$

x_v^* : $<1/2$ $\geq 1/2$ $\geq 1/2$

$$|S_{LP}| \leq 2 \cdot \sum_{v \in V} x_v^* \to ②$$

by construction.

x^* is LP optimum. Suppose y^* is ILP optimum

$$\sum_{v \in V} x_v^* \leq \sum_{v \in V} y_v^* = |S_{ILP}| \longrightarrow ①$$

$|S_{LP}| = \langle 0, 1, 1, 0, 1 \ldots 0 \rangle$ $\quad |S_{LP}| \leq 2 \cdot \sum_{v \in V} x_v^*$ $\to ②$

x_v^* : $\quad <1/2 \quad \geq 1/2 \quad \geq 1/2$ $\quad$ by construction.

From ① and ② :

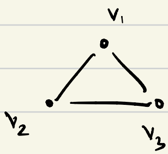
$$|S_{LP}| \leq 2 \cdot \sum_{v \in V} x_v^* \leq 2 \cdot \sum_{v \in V} y_v^* = 2\,|S_{ILP}|$$

$\Rightarrow$ $|S_{LP}| \leq 2 \cdot |\text{min vertex cover}|$.

Maximum Independent Set

Given a graph $G = (V, E)$

Independent set: $S \subseteq V$ s.t. there is no edge between vertices in S.



$\{v_1\}$ $\{v_2\}$, $\{v_3\}$

Goal: find an independent set with the maximum no. of vertices.

ILP:

$$\text{maximize} \sum_{v \in V} x_v$$

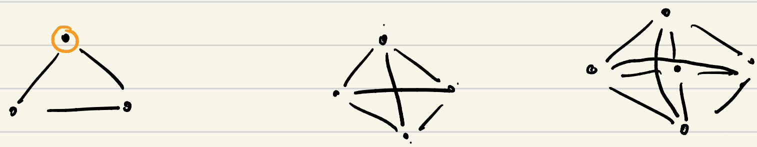
$$\text{s.t.} \quad x_u + x_v \leq 1 \quad \forall (u, v) \in E$$

$$x_v \in \{0, 1\}$$

LP relaxation: $0 \leq x_v \leq 1$

Consider the family of complete graphs:

K_n: complete graph with n vertices.



ILP optimum: 1 1 1

LP optimum: $3/2$ $4/2$ $5/2$

K_n: ILP optimum $= 1$

LP optimum $= n/2$

LP relaxation does not give useful information about the ILP optimum.

Summary:

1. LP optimum = ILP optimum (Max bipartite matching / Min bipartite vertex cover . TUM)

2. LP optimum gives constant approximation (Min. Vertex cover) general

3. LP optimum gives no useful information (max independent set)