Linear Programming & Combinatorial Optimization

Lecture 11

Agenda:

- 1. Proof of Farkas' lemma
- 2. An application of LP : Zero-sum games

PROOF OF FARKAS' LEMMA:

(iii) $Ax \leq b$ has a soln. iff for every non-negative $y \in \mathbb{R}^m$ s.t. $y^T A = 0$, we have $y^T b \geq 0$

Proof: ($\Rightarrow$) Suppose $\tilde{x}$ is a soln. to $Ax \leq b$.

Pick some y s.t. $y^T A = 0$

$$A\tilde{x} \leq b$$

$$\underbrace{y^T A}\, \tilde{x} \leq y^T b$$

$$0 \leq y^T b$$

($\Leftarrow$) is not immediate.

FOURIER-MOTZKIN METHOD:

$$\boxed{\begin{aligned} 5x + 4y + 2z &\leq 15 \\ 2x + 2y - 3z &\leq 10 \\ -x + y + 4z &\leq 20 \end{aligned}}$$

$$\downarrow$$

Get a system with 'y' and 'z' s.t. original system has a soln. iff. the new system has a soln.

With a single var:

$$\left.\begin{array}{lll} z & \le & 4 \\ z & \le & 5 \end{array}\right\} \Rightarrow z \le 4$$

$$\left.\begin{array}{lll} -z & \le & 2 \\ -z & \le & -1 \end{array}\right\} \Rightarrow z \ge 1$$

$z \ge -2$

$z \ge 1$

$-2 \le 4$

$-2 \le 5$

$1 \le 4$

$1 \le 5$

Now: when there are multiple variables:

$$\begin{array}{rcrcrcl} 5x & + & 4y & + & 2z & \le & 15 \\ 2x & + & 2y & - & 3z & \le & 10 \\ -x & + & y & + & 4z & \le & 20 \\ -x & + & 2y & - & 2z & \le & 15 \end{array}$$

$10 \le 5$

$\boxed{0 \le -1}$

$x + \frac{4}{5}y + \frac{2z}{5} \le 3$

$-x + y + 4z \le 20$

$\left(\frac{4}{5}+1\right)y + \left(\frac{2}{5}+4\right)z \le 3+20$

$$x \le \frac{15}{5} - \frac{4y}{5} - \frac{2z}{5}$$

$$x \le \frac{10}{2} - \frac{2y}{2} - \frac{3z}{2}$$

give upper bounds on x

$$-x \le 20 - y - 4z$$

$$x \ge -20 + y + 4z$$

$$-x \le 15 - 2y + 2z$$

$$x \ge -15 + 2y - 2z$$

give lower bounds.

We will generate a system which declares that:

max(lower bounds of x) $\leq$ min (upper bounds for x)

$$x \leq \frac{15}{5} - \frac{4y}{5} - \frac{2z}{5}$$

$$x \leq \frac{10}{2} - \frac{2y}{2} - \frac{3z}{2}$$

give upper bounds on x

$$x \geq -20 + y + 4z$$

$$x \geq -15 + 2y - 2z$$

give lower bounds.

$(\frac{4}{5}+1)y + (\frac{2}{5}+4)z \leq 3+20$

$$\boxed{\begin{aligned}
-20 + y + 4z &\leq \frac{15}{5} - \frac{4y}{5} - \frac{2z}{5} \\
-20 + y + 4z &\leq \frac{10}{2} - \frac{2y}{2} - \frac{3}{2}z \\
-15 + 2y - 2z &\leq \frac{15}{5} - \frac{4y}{5} - \frac{2z}{5} \\
-15 + 2y - 2z &\leq \frac{10}{2} - \frac{2y}{2} - \frac{3}{2}z
\end{aligned}}$$

↳ new system with one variable less.

Original system has a soln. iff new system has a soln.

Proof of Farkas' lemma: Suppose $Ax \leq b$ has no soln.

Induction on the number of variables 'n'.

Base case: $n = 0$

y:	
0	$4 \leq 5$
0	$-1 \leq 2$
0	$\vdots$
$\vdots$	
1	$0 \leq -1$ since $Ax \leq b$ has no soln.
0	$2 \leq 3$
$\vdots$	$\vdots$
0	$14 \leq 20$

What is A?

$n = 1$:

y:		
0	$x_1 \leq 4$	
0	$x_1 \leq 5$	
1	$x_1 \leq 3$	i
0	$x_1 \leq 7$	
0	$-x_1 \leq -1$	
$\vdots$	$-x_1 \leq -2$	
$\vdots$	$\vdots$	
0	$-x_1 \leq 3$	
1	$-x_1 \leq -4$	j

$$A = \begin{array}{c} \left[\begin{array}{c} 1 \\ \vdots \\ 1 \\ 1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{array}\right] \end{array} \begin{array}{c} \\ \\ i \\ \\ \\ j \\ \\ \\ \end{array}$$

$$y = \begin{array}{c} [\,0\ 0 \cdots \underset{i}{1}\ 0 \cdots \underset{j}{1} \cdots 0\,] \end{array}$$

$y^T A = 0$

$y^T b = -1$ in this example.

In general, such a y will give $y^T A = 0$ and $y^T b < 0$

Induction case:

$$Ax \leq b \qquad x: \ n \text{ variables}$$

$\downarrow$ Eliminate some variable using Fourier Motzkin

$$A'\tilde{x} \leq b' \qquad \tilde{x}: \ n-1 \text{ variables}$$

Since $Ax \leq b$ has no solution, we know:
$A'\tilde{x} \leq b'$ has no solution.

Apply induction hypothesis.

$\exists \ \tilde{y}$ s.t. $\tilde{y} A' = 0$ and $\tilde{y} b < 0$

Now we need to generate y for $Ax \leq b$:

$$\begin{array}{ccccccccc} & & & & i & & j & & \\ [0 & 0 & \cdot & \cdots & 1/a & 0 & \cdots & 1/a' & 0 \ 0 \ 0] \end{array}$$

$(\ . \quad . \quad . \quad . \ )$

A

$$\begin{bmatrix} a x + \cdots\cdots \leq b_i \\ \\ -a'x + \cdots\cdots \leq b_j \\ \end{bmatrix} \qquad a, a' \geq 0$$

A' is obtained as some: MA

b' : Mb

$$A' = M A$$
$$b' = M b$$

$$\tilde{y} A' = 0 \qquad \Rightarrow \qquad \tilde{y} M A = 0$$
$$\tilde{y} b' < 0 \qquad\qquad\qquad \tilde{y} M b < 0$$

Pick $y^T = \tilde{y}^T M$

$A: m \times n$

$M: [m' \times m]$

Then $y A = 0$

$y b < 0$

$y = m \times 1$

$\tilde{y} = m' \times 1$

$\tilde{y}^T M = 1 \times m$

Summary: - Proof of Farkas' lemma
- Proof of duality via Farkas' lemma.

APPLICATIONS OF LINEAR PROGRAMMING: ZERO - SUM GAMES

REFERENCE: Section 8.1 of text:

Understanding and Using Linear Programming
— Matoušek & Gärtner

ZERO - SUM GAMES:

Two players: Maximizer Vs Minimizer

- Maximizer has m - strategies $\{1, 2, 3, \dots, m\}$
- Minimizer has n - strategies $\{1, 2, 3, \dots, n\}$
- An $m \times n$ payoff matrix M is given

Example:

Min chooses columns

Max chooses Rows

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 10 & 0 & -1 \\ 2 & -2 & 4 & 0 \\ 3 & 5 & 3 & 1 \\ 4 & 7 & 2 & -2 \\ 5 & 4 & -1 & 1 \end{array} \longrightarrow M : \text{Payoff matrix}$$

- When Max plays i and Min plays j,
  Payoff $= m_{ij}$

Max receives m_{ij} from Min

OBJECTIVE / GOAL OF THE GAME:

- Maximizer wants to maximize the payoff
- Minimizer wants to minimize the payoff

Zero-sum: one's loss is the other's gain

- A payoff of 5 to Max is −5 to Min
- A payoff of −3 to Max is +3 to Min

NOTE: Many situations in economics / AI / finance involving strategic reasoning can be modeled as zero-sum games.

SOME EXAMPLES:

I)

	1	2	3
1	10	0	−1
2	−2	4	0
3	5	3	1
4	7	2	−2
5	4	−1	1

When Max plays 1, best strategy of Min is 3, Payoff = −1

Max plays	best strategy of Min	Payoff
1	3	−1
2	1	−2
3	3	1
4	3	−2
5	2	−1

$\boxed{\text{max min} = 1}$

When Min plays 1, best strategy of Max is 1, Payoff = 10

Min plays	best strategy of Max	Payoff
1	1	10
2	2	4
3	3/5	1

$\boxed{\text{min max} = 1}$

II)

	1	2	3
1	10	0	−1
2	−2	4	0
3	5	3	7
4	7	2	−2
5	4	−1	1

When Max plays	best strategy of Min is	Payoff =
1	3	−1
2	1	−2
3	2	3
4	3	−2
5	2	−1

$\boxed{\text{max min} = 3}$

When Min plays	best strategy of Max is	Payoff =
1	1	10
2	2	4
3	3	7

$\boxed{\text{min max} = 4}$

Remarks:

max min:
- Max plays first.
- Knowing Max's strategy Min gives her best response
- Knowing that Min will play best response, Max plays a strategy that maximizes the payoff.

min max
- Min plays first
- Max gives her best response
- Min plays a strategy that minimizes the payoff, knowing that Max will play best response.

More formally:

Given a game represented by payoff matrix M:

$$\text{max-min-pure}(M) = \max_{i \in \{1,\dots,m\}} \ \min_{j \in \{1,2,\dots,n\}} m_{ij}$$

$$\text{min-max-pure}(M) = \min_{j \in \{1,\dots,n\}} \ \max_{i \in \{1,\dots,m\}} m_{ij}$$

Pure strategies:
- The choices $1,\dots m$ for Max are called her pure / deterministic strategies
- Choices $1 \dots n$ of Min are called her pure / deterministic strategies

Later we will see other kinds of strategies

Lemma: $\text{max-min-pure}(M) \leq \text{min-max-pure}(M) \quad \forall M$

Proof: For each $i \in \{1,2,\dots m\}$ (a pure strategy of Max):

$$\min_{j \in \{1,2,\dots n\}} m_{ij} \leq \min_j \max_i m_{ij}$$

Hence $\max_i \min_j m_{ij} \leq \min_j \max_i m_{ij}$ $\square$

Saddle points:

We saw that max-min-pure$(M) \leq$ min-max-pure(M)

In some cases, max-min-pure = min-max-pure

(3, 3)

$$
\begin{array}{c|ccc|}
 & 1 & 2 & 3 \\
\hline
1 & 10 & 0 & -1 \\
2 & -2 & 4 & 0 \\
3 & 5 & 3 & 1 \\
4 & 7 & 2 & -2 \\
5 & 4 & -1 & 1 \\
\hline
\end{array}
\longrightarrow
$$

Least in row

Greatest in column

m_{kl} is a saddle point if $m_{kl} = \min\limits_{j} m_{kj}$

$$= \max_{i} m_{il}$$

- max-min-pure = min-max-pure iff there is a saddle point.

Prove this statement: Exercise.

NASH EQUILIBRIUM (over pure strategies):

When max-min-pure (M) = min-max-pure (M)

the game is said to have a $\boxed{\text{Nash equilibrium over pure strategies}}$

- In this case, there exist strategies i, j for Max / Min s.t.

i) j is the best response of Min to i

and

ii) i is the best response of Max to j

	1	2	3
1	10	0	-1
2	-2	4	0
3	5	3	1
4	7	2	-2
5	4	-1	1

→ Least in row

Greatest in column

Nash equilibrium strategies: 3 for Max

3 for Min

- Max does not gain anything by deviating
- Similarly, Min does not have an incentive to deviate.

Summary:

- 1. Zero-sum games, Pure strategies
- 2. max-min-pure and min-max-pure
- 3. In general: max-min-pure $\leq$ min-max-pure
- 4. Nash equilibrium: when max-min-pure = min-max-pure.

Where is Linear Programming in all this?

- Next : different kind of strategies,
  - ↳ max min and min max over such strategies
  - ↳ significant use of LP.

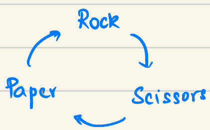
ZERO-SUM GAMES - Part 2

Last part: max min, min max over pure strategies

This part: Mixed / randomized strategies.

Example:

$$\text{Max} \left\{ \begin{array}{c|ccc} & \overbrace{\text{Rock} \quad \text{Paper} \quad \text{Scissors}}^{\text{Min}} & & \\ \text{Rock} & 0 & -1 & +1 \\ \text{Paper} & +1 & 0 & -1 \\ \text{Scissors} & -1 & +1 & 0 \end{array} \right. \qquad M_{RPS}$$



↳ Payoff for Maximizer

$\text{max-min-pure}(M_{RPS}) = -1$

$\text{min-max-pure}(M_{RPS}) = +1$

Consider a different strategy for Max:

- Max plays each strategy with 1/3 Probability

Mixed Strategy

$$\boxed{\sigma : \quad \tfrac{1}{3}\,\text{Rock} + \tfrac{1}{3}\,\text{Paper} + \tfrac{1}{3}\,\text{Scissors}}$$

If Min plays pure strategy Rock, "Expected Payoff" $= \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot (-1)$

Paper, $= \frac{1}{3}(-1) + \frac{1}{3}(0) + \frac{1}{3} \cdot 1$

Scissors, $= \frac{1}{3} \cdot 1 + \frac{1}{3}(-1) + \frac{1}{3}(0)$

When Max plays mixed strategy σ and Min plays some pure strategy, the expected payoff $= 0$

- But Min can also play mixed strategies. Suppose Min plays:

$$\boxed{\tau : \quad \tfrac{1}{3}\text{ Rock} + \tfrac{1}{3}\text{ Paper} + \tfrac{1}{3}\text{ Scissors}}$$

- What is the expected payoff when σ and τ are played?

$$\begin{array}{cc|ccc} & & \text{Rock } \tfrac{1}{3} & \text{Paper } \tfrac{1}{3} & \text{Scissors } \tfrac{1}{3} \\ \hline \tfrac{1}{3} & \text{Rock} & 0 & -1 & +1 \\ \tfrac{1}{3} & \text{Paper} & +1 & 0 & -1 \\ \tfrac{1}{3} & \text{Scissors} & -1 & +1 & 0 \end{array}$$

$$\tfrac{1}{9}\left[\underbrace{(0 - 1 + 1)}_{\text{Row 1}} + \underbrace{(1 + 0 - 1)}_{\text{Row 2}}\underbrace{(-1 + 1 + 0)}_{\text{Row 3}}\right] = 0$$

Mixed Strategy: is a probability distribution over pure strategies.

Maximizer's pure strategies : $\{1, 2, 3, \ldots, m\}$

Mixed strategy for Max : $x_1, x_2, \ldots, x_m \in \mathbb{R}_{\geq 0}$ $\in [0,1]$

$$\text{s.t. } \sum_{i=1}^{m} x_i = 1$$

Minimizer's pure strategies : $\{1, 2, 3, \ldots, n\}$

Mixed strategy for Mi : $y_1, y_2, \ldots y_n \in \mathbb{R}_{\geq 0}$ $\in [0,1]$

$$\text{s.t. } \sum_{j=1}^{n} y_j = 1$$

Payoff: Given $\sigma := (x_1, x_2, \dots, x_m)$ and $\tau := (y_1, y_2, \dots, y_n)$

$$\boxed{\text{Payoff}(\sigma,\tau) = \sum_{i \in \{1,\dots m\}} \sum_{j \in \{1 \dots n\}} x_i \cdot y_j \cdot m_{ij}}$$

$$= x^T M y$$

We are interested in the following quantities:

$$\max\min(M) = \max_{\substack{\text{mixed strategies } \sigma \\ \text{of Max}}} \min_{\substack{\text{mixed strategies } \tau \\ \text{of Min}}} \text{Payoff}(\sigma,\tau)$$

$$\min\max(M) = \min_{\substack{\text{mixed strategies } \tau \\ \text{of Min}}} \max_{\substack{\text{mixed strategies } \sigma \\ \text{of Max}}} \text{Payoff}(\sigma,\tau)$$

Notice that there are infinitely many mixed strategies.

Goal: Given M, how do we compute

$\max\min(M)$?

- This does not look like a linear optimization problem since there is a mix of objectives and the cost that needs to be optimized: $x^T M y$ has bilinear terms

- We will now see that computing $\max\min(M)$ can be reduced to an LP problem.

1. Suppose we fix a strategy $\langle x_1, x_2, \dots, x_m \rangle$ for Max

- For example : $\langle 1/3, 1/3, 1/3 \rangle$ in M_{RPS} .

2. Given the strategy $\langle x_1, \dots x_m \rangle$ we want to find:

$$\min_{y} \quad x^T M y$$

mixed strategies of Min

3. This can be written as a linear program (as x_i's are constants)

$$\boxed{\begin{array}{ll} \text{minimize} & x^T M y \\ \text{subject to} & y_1 + y_2 + \dots + y_n = 1 \\ & y_1, y_2, \dots, y_n \geq 0 \end{array}} \qquad (*)$$

For example, with $\langle 1/3, 1/3, 1/3 \rangle$ in RPS:

$$\text{minimize} \quad -1/3\, y_2 + 1/3\, y_3 + 1/3\, y_1 - 1/3\, y_3 - 1/3\, y_1 + 1/3\, y_2$$

$$\text{subject to} \quad y_1 + y_2 + y_3 = 1$$
$$y_1, \dots, y_3 \geq 0$$

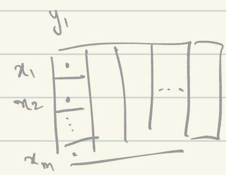
4. The answer to above LP gives the best response of Min to Max's mixed strategy $\langle x_1, x_2, \dots, x_m \rangle$

Each $\langle x_1, x_2, \dots x_m \rangle \longrightarrow x_0$ (answer of above LP)

We want the maximum possible x_0 .

5. Let us first write the dual of (*):

$$\boxed{\begin{aligned} &\text{minimize} && x^T M y \\ &\text{subject to} && y_1 + y_2 + \cdots + y_n = 1 \\ & && y_1, y_2, \ldots, y_n \geq 0 \end{aligned}}$$



min. $b^T y$
$A^T y = c$
$y \geq 0$ $\leftarrow$ max $c^T x$
$Ax \leq b$

$$\begin{aligned} &\text{maximize} && x_0 \\ &\text{subject to:} && x_0 \leq M_{11} x_1 + M_{21} x_2 + \cdots + M_{m1} x_m \\ & && x_0 \leq M_{12} x_1 + M_{22} x_2 + \cdots + M_{m2} x_m \\ & && \vdots \\ & && x_0 \leq M_{1n} x_1 + M_{2n} x_2 + \cdots + M_{mn} x_m \end{aligned}$$

$$\boxed{\begin{aligned} &\text{maximize} && x_0 \\ &\text{subject to} && \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{n \times 1} x_0 \leq M^T x \end{aligned}} \qquad (**)$$

n rows $\rightarrow$

6. Primal is bounded (feasible region is contained in unit hypercube)

Hence: Optimum of primal = optimum of dual

– For a given strategy of Max $\langle x_1, x_2, \ldots, x_m \rangle$, the payoff obtained when Min plays her best response is given by the optimum cost of LP $(**)$.

7. We want to maximize the optimum of LP $(**)$ w.r.t. all mixed strategies of Maximizer.

- Hence we consider $\langle x_1, x_2, \ldots, x_m \rangle$ as variables and add the constraint $\sum_{i=1}^{m} x_i = 1$ to $(**)$

$$\boxed{\begin{array}{ll} \text{maximize} & x_0 \\ \text{Subject to} & \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} x_0 \leq M^T x \\ & x_1 + x_2 + \cdots + x_m = 1 \\ & x_1, x_2, \ldots, x_m \geq 0 \end{array}}$$

LP for max min (M) [over mixed strategies]

Illustration on the RPS example:

	Rock	Paper	Scissors
Rock	0	-1	+1
Paper	+1	0	-1
Scissors	-1	+1	0

maximize x_0

subject to:

$$x_0 \leq 0 \cdot x_1 + 1 \cdot x_2 - 1 \cdot x_3$$
$$x_0 \leq -1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3$$
$$x_0 \leq +1 \cdot x_1 - 1 \cdot x_2 + 0 \cdot x_3$$

$$x_1 + x_2 + x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

$\leftarrow$ LP to find max min.

APPLICATIONS OF LP: ZERO-SUM GAMES - Part 3:

Recall:

- For pure strategies, max min $\leq$ min max, and equality is attained iff there are saddle points (Part 1)

- Mixed strategies, and an LP to compute max min over mixed strategies. (Part 2)

Today: We will prove that over mixed strategies: max min = min max.

LP for max min:

maximize x_0

subject to:

$$x_0 \leq 0 \cdot x_1 + 1 \cdot x_2 - 1 \cdot x_3$$
$$x_0 \leq -1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3$$
$$x_0 \leq +1 \cdot x_1 - 1 \cdot x_2 + 0 \cdot x_3$$
$$x_1 + x_2 + x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

		y_1 Rock	y_2 Paper	y_3 Scissors
x_1	Rock	0	-1	+1
x_2	Paper	+1	0	-1
x_3	Scissors	-1	+1	0

LP for min max:

minimize y_0

subject to:

$$0 \cdot y_1 - 1 \cdot y_2 + 1 \cdot y_3 \leq y_0$$
$$1 \cdot y_1 + 0\, y_2 - 1 \cdot y_3 \leq y_0$$
$$-1\, y_1 + 1\, y_2 + 0\, y_3 \leq y_0$$
$$y_1 + y_2 + y_3 = 1$$
$$y_1, y_2, y_3 \geq 0$$

Dual of the LP for max min:

	Rock	Paper	Scissors
Rock	0	−1	+1
Paper	+1	0	−1
Scissors	−1	+1	0

maximize x_0

subject to:

$$
\begin{aligned}
y_1 \times \quad x_0 &\le 0\cdot x_1 + 1\cdot x_2 - 1\cdot x_3 \\
y_2 \times \quad x_0 &\le -1\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 \\
y_3 \times \quad x_0 &\le +1\cdot x_1 - 1\cdot x_2 + 0\cdot x_3
\end{aligned}
$$

$$
\begin{aligned}
y_0 \times \quad x_1 + x_2 + x_3 &= 1 \\
x_1, x_2, x_3 &\ge 0
\end{aligned}
$$

minimize y_0

subject to:

$$
y_1 + y_2 + y_3 = 1 \qquad (x_0)
$$

$$
\begin{aligned}
0\cdot y_1 + 1\cdot y_2 - 1\cdot y_3 + y_0 &\ge 0 \qquad (x_1) \\
-1\cdot y_1 + 0\cdot y_2 + 1\cdot y_3 + y_0 &\ge 0 \qquad (x_2) \\
+1\cdot y_1 - 1\cdot y_2 + 0\cdot y_3 + y_0 &\ge 0 \qquad (x_3)
\end{aligned}
$$

$$
y_1, y_2, y_3 \ge 0 \qquad (\text{inequalities in primal})
$$

This is exactly the LP for min-max!

- Since primal is bounded and feasible, there is an optimum.

Hence primal-optimum = dual-optimum, i.e.. max min = min max!

In general: Given game M

max-min LP:

maximize x_0

subject to:

$$x_0 \le M_{11} x_1 + M_{21} x_2 + \dots + M_{m1} x_m$$
$$x_0 \le M_{12} x_1 + M_{22} x_2 + \dots + M_{m2} x_m$$
$$\vdots$$
$$x_0 \le M_{1n} x_1 + M_{2n} x_2 + \dots + M_{mn} x_m$$

$$x_1 + x_2 + \dots + x_m = 1$$
$$x_1, x_2, \dots, x_m \ge 0$$

min-max LP:

minimize y_0

subject to:

$$y_0 \ge M_{11} y_1 + M_{12} y_2 + \dots + M_{1n} y_n$$
$$y_0 \ge M_{21} y_1 + M_{22} y_2 + \dots + M_{2n} y_n$$
$$\vdots$$
$$y_0 \ge M_{m1} y_1 + M_{m2} y_2 + \dots + M_{mn} y_n$$

$$y_1 + y_2 + \dots + y_n = 1$$
$$y_1, y_2, \dots, y_n \ge 0$$

These two LPs are duals of each other.

Minimax theorem: max-min = min-max over mixed strategies

There exist mixed strategies $\bar{x}$ and $\bar{y}$ for Max and Min s.t.:

- $\bar{y}$ is the best response to $\bar{x}$ and
- $\bar{x}$ is the best response to $\bar{y}$

There is a Nash equilibrium over mixed strategies for zero-sum games.
