Linear Programming & Combinatorial Optimization

Lecture 10

PROOF OF DUALITY VIA FARKAS' LEMMA

REFERENCE: Sections 6.4 & 6.7 of text:

Understanding and Using Linear Programming
— Matoušek & Gärtner

FARKAS' LEMMA:

Let A be an $m \times n$ real matrix and let $b \in \mathbb{R}^m$.
Then exactly one of the following holds:

1. There is a solution to $Ax = b$, $x \geq 0$.
2. There is a $y \in \mathbb{R}^m$ s.t. $y^T A \geq 0$ and $y^T b < 0$.

$Ax = b$ has a non-negative solution iff

for every $y \in \mathbb{R}^m$ s.t. $y^T A \geq 0$, we have $y^T b \geq 0$

VARIANT OF FARKAS' LEMMA:

Let A be an $m \times n$ real matrix, let $b \in \mathbb{R}^m$.

(i) $Ax = b$ has a non-negative soln. iff for every $y \in \mathbb{R}^m$ s.t. $y^T A \geq 0$, we have $y^T b \geq 0$.

(ii) $Ax \leq b$ has a non-negative soln. iff for every non-negative $y \in \mathbb{R}^m$ s.t. $y^T A \geq 0$, we have $y^T b \geq 0$

(iii) $Ax \leq b$ has a soln. iff for every non-negative $y \in \mathbb{R}^m$ s.t. $y^T A = 0$, we have $y^T b \geq 0$

Exercise: Prove that (i), (ii), (iii) are equivalent.

PROOF OF DUALITY:

Primal: (P)

maximize $c^T x$

$$Ax \leq b$$
$$x \geq 0$$

Dual (D):

minimize $b^T y$

$$A^T y \geq c$$
$$y \geq 0$$

Strong duality: If (P) has an optimum, then (D) is feasible, has an optimum, and the optimal costs coincide.

Suppose (P) has optimum at $x^* \in \mathbb{R}^n$.

Let $\gamma = c^T x^*$

Consider the system:

$$\boxed{\begin{aligned} Ax &\leq b \\ c^T x &\geq \gamma \end{aligned}}$$

→ x^* is a soln. to this system.

$$\boxed{\begin{aligned} Ax &\leq b \\ c^T x &\geq \gamma \end{aligned}}$$

$\rightarrow$ x^* is a non-negative soln. to this system.

$$\boxed{\begin{aligned} Ax &\leq b \\ c^T x &\geq \gamma + \epsilon \end{aligned}}$$

$\rightarrow$ no non-negative soln.

$\epsilon > 0$

(ii) $Ax \leq b$ has a non-negative soln. iff for every non-negative $y \in \mathbb{R}^m$ s.t. $y^T A \geq 0$, we have $y^T b \geq 0$

$$Ax \leq b$$

$$-c^T x \leq -\gamma - \epsilon$$

<u>$\epsilon > 0$:</u>

$$A' = \begin{bmatrix} A \\ -c^T \end{bmatrix} \qquad b' = \begin{bmatrix} b \\ -\gamma - \epsilon \end{bmatrix}$$

From (ii) There exists non-negative $y \in \mathbb{R}^{m+1}$ s.t. $y^T A' \geq 0$

$y^T b' < 0$

$$Ax \leq b$$

$$-c^T x \leq -\gamma - \epsilon$$

$\underline{\epsilon > 0:}$

$$A' = \begin{bmatrix} A \\ -c^T \end{bmatrix} \qquad b' = \begin{bmatrix} b \\ -\gamma - \epsilon \end{bmatrix}$$

From (ii) There exists non-negative $y \in \mathbb{R}^{m+1}$ s.t. $y^T A' \geq 0$

$y^T b' < 0$

$$y = \begin{bmatrix} \left.\begin{matrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{matrix}\right\} u \\ z \end{bmatrix}$$

$$y^T A' \geq 0 \iff u^T A \geq zc$$

$$\underline{[u_1 \dots u_m} \;\; z] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ & \vdots & & \\ a_{m1} & \cdots & & a_{mn} \\ -c_1 & \cdots & & c_n \end{bmatrix} \geq \left.\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} n \text{ dimensional}$$

$$[u_1 \dots u_m] \begin{bmatrix} A \end{bmatrix} \geq z \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$u^T A \geq zc \qquad z \geq 0$$

$$b' = \begin{bmatrix} b \\ -\gamma - \epsilon \end{bmatrix}$$

$$[u_1 \dots \quad u_m \quad z] \begin{bmatrix} b \\ -\gamma - \epsilon \end{bmatrix} < 0$$

$$u_1 b_1 + \dots + u_m b_m < z(\gamma + \epsilon)$$

$$b^T u < z(\gamma + \epsilon) \longrightarrow (*)$$

$$\rightarrow u^T A \geq zc$$

From 'u' we want to generate a feasible soln. to the dual (D).

Natural option would be to consider $\frac{u}{z}$. But we can do this only when $z > 0$, We will prove that $z > 0$.

$$Ax \leq b \qquad (\epsilon = 0)$$
$$-c^T x \leq -\gamma$$

This has a soln. $\therefore$ From (ii),

since $[u_1 \ldots u_m \ z] \begin{bmatrix} A \\ -c \end{bmatrix} \geq 0$,

we should have:

$$[u_1 \ldots u_m \ z] \begin{bmatrix} b \\ -\gamma \end{bmatrix} \geq 0$$

$$b^T u \geq z\gamma \longrightarrow (\#)$$

From (*) and (#) $z > 0$ (otherwise

(*) says $b^T u < 0$

(#) says $b^T u \geq 0$.

Coming back to:

$$b^T u < z(\gamma + \epsilon)$$

$$u^T A \geqslant zc$$

$$v = \frac{1}{z} \cdot u$$

We have: $$b^T v < \gamma + \epsilon$$

$$v^T A \geqslant c$$

$\therefore$ v is a feasible soln. to the dual.

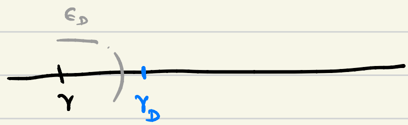
Moreover v has cost $< \gamma + \epsilon$.

From weak duality, cost $(v) = b^T v \geqslant \gamma$.

$$\gamma \leqslant b^T v < \gamma + \epsilon \longrightarrow (*)$$

→ Since (D) is feasible and bounded, there exists an optimum.

→ From $(*)$, for every ϵ, there exists a dual soln. with cost between $[\gamma, \gamma+\epsilon)$



Suppose dual optimum is some γ_D with $\gamma < \gamma_D$,
then we can find an ϵ_D s.t. there exists
a dual soln. with cost in $[\gamma, \gamma + \epsilon_D) < \gamma_D$.

→ This is a contradiction.

→ Hence, (D) optimum equals γ.

PROOF OF FARKAS' LEMMA:

(iii) $Ax \leq b$ has a soln. iff for every non-negative $y \in \mathbb{R}^m$ s.t. $y^T A = 0$, we have $y^T b \geq 0$

Proof: ($\Rightarrow$) Suppose $\tilde{x}$ is a soln. to $Ax \leq b$.

Pick some y s.t. $y^T A = 0$

$$A\tilde{x} \leq b$$

$$\underbrace{y^T A}\,\tilde{x} \leq y^T b$$

$$0 \leq y^T b$$

($\Leftarrow$) is not immediate.

FOURIER-MOTZKIN METHOD:

$$\boxed{\begin{aligned} 5x + 4y + 2z &\leq 15 \\ 2x + 2y - 3z &\leq 10 \\ -x + y + 4z &\leq 20 \end{aligned}}$$

$$\downarrow$$

Get a system with 'y' and 'z' s.t. original system has a soln. iff the new system has a soln.

With a single var:

$$\left.\begin{array}{ll} z & \leq 4 \\ z & \leq 5 \end{array}\right\} \Rightarrow z \leq 4$$

$$\left.\begin{array}{ll} -z & \leq 2 \\ -z & \leq -1 \end{array}\right\} \Rightarrow z \geq 1$$

Now: when there are multiple variables:

$$\begin{array}{rcl} 5x + 4y + 2z & \leq & 15 \\ 2x + 2y - 3z & \leq & 10 \\ -x + y + 4z & \leq & 20 \\ -x + 2y - 2z & \leq & 15 \end{array}$$

$$x \leq \frac{15}{5} - \frac{4y}{5} - \frac{2z}{5}$$

$$x \leq \frac{10}{2} - \frac{2y}{2} - \frac{3z}{2}$$

give upper bounds on x

$$-x \leq 20 - y - 4z$$

$$x \geq -20 + y + 4z$$

$$-x \leq 15 - 2y + 2z$$

$$x \geq -15 + 2y - 2z$$

give lower bounds.

We will generate a system which declares that:

max (lower bounds of x) $\leq$ min (upper bounds for x)

$$x \leq \frac{15}{5} - \frac{4y}{5} - \frac{2z}{5}$$

$$x \leq \frac{10}{2} - \frac{2y}{2} - \frac{3z}{2}$$

give upper bounds on x

$$x \geq -20 + y + 4z$$

$$x \geq -15 + 2y - 2z$$

give lower bounds.

$$\boxed{\begin{aligned} -20 + y + 4z &\leq \frac{15}{5} - \frac{4y}{5} - \frac{2z}{5} \\ -20 + y + 4z &\leq \frac{10}{2} - \frac{2y}{2} - \frac{3}{2}z \\ -15 + 2y - 2z &\leq \frac{15}{5} - \frac{4y}{5} - \frac{2z}{5} \\ -15 + 2y - 2z &\leq \frac{10}{2} - \frac{2y}{2} - \frac{3}{2}z \end{aligned}}$$

↳ new system with one variable less.

Original system has a soln. iff new system has a soln.