Lecture 1: Introduction to Linear Programs

Today's lecture:

- 1. Examples of optimization problems
- 2. Linear Programming problem
- 3. Different forms of LPs.

This course:

Prerequisites: Algorithms, Basic linear algebra

Evaluation: 2 Quizzes (30%)

+ Midsem (30%)

+ Endsem (40%)

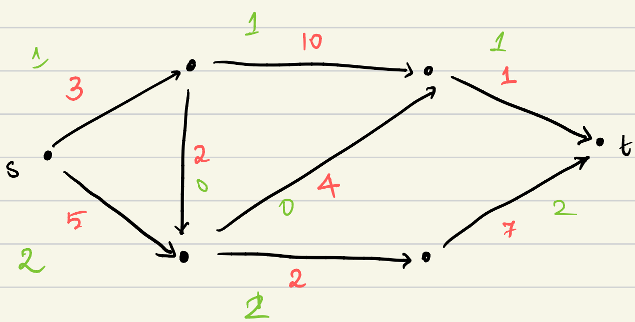
Exam dates appear in the course web page.

Example 1: Network flow

Network $N = (s, t, V, E, b)$ is a directed graph.

source, target, Vertices, Edges, Bound / capacity

$b: E \longmapsto \mathbb{N}$



Flow: A flow f is a function:

$$f: E \longmapsto \mathbb{R}^{\geq 0}$$

s.t.

1. $f(u, v) \leq b(u, v)$

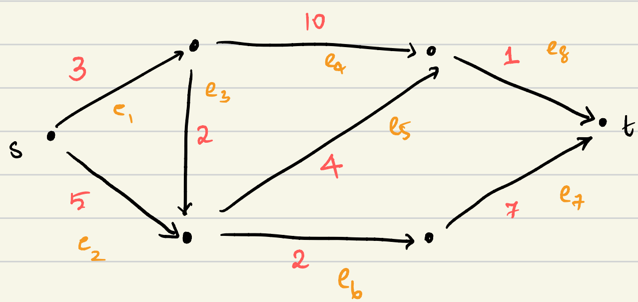
2. $\sum_{(u,v) \in E} f(u, v) = \sum_{(v,u) \in E} f(v, u) \quad \forall v \in V - \{s, t\}$

Max-flow problem:

Find a flow f s.t.

$$\sum_{s \to v} f(s, v) \quad \text{is maximum}$$

Max-flow as a linear Program:



Variables: x_{e_i} $\quad \forall i \in \{1, 2, \ldots 8\}$ Real variables

Objective function

$$\text{Maximize} \quad \sum_{(s,u) \in E} x_{(s,u)} \quad \to \quad x_{e_1} + x_{e_2}$$

Constraints

$$x_{e_i} \leq b_{e_i}$$

$\forall v \in V - \{s, t\}$

$$\sum_{(u,v) \in E} x_{(u,v)} = \sum_{(v,w) \in E} x_{(v,w)} \quad \to \quad x_{e_3} + x_{e_2} = x_{e_6} + x_{e_5}$$

$$x_{e_i} \geq 0$$

Example 2: Diet problem

n foods

Each food has some amount of 'm' nutrients.

a_{ij} : amount of i^{th} nutrient in 1 unit of j^{th} food

$i = 1, 2, \dots m, \qquad j = 1, 2, \dots n$

r_i : yearly requirement of i^{th} nutrient

c_j: cost per unit of j^{th} food.

Find least expensive diet that is nutritionally adequate

LP for the diet problem:

Variables: $x_j \quad \forall i \in \{1, \dots, n\}$

Constraints: $a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \geq r_1$

$$\vdots$$

$$\sum_{j=1}^{n} a_{ij} x_j \geq r_i$$

$$x_j \geq 0 \qquad \forall j$$

Objective: minimize $\sum_{j=1}^{n} c_j x_j$

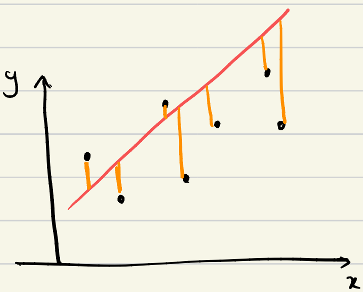
Example 3: Fitting a line

Given a set of 'n' points

(x_1, y_1) (x_2, y_2) $, \dots \dots ,$ (x_n, y_n)

Find a line $y = ax + b$ that minimizes the total error:

$$\sum_{i=1}^{n} | a x_i + b - y_i |$$



Error for the i^{th} point: $|ax_i + b - y_i|$

$$\text{minimize} \quad \sum_{i=1}^{n} |ax_i + b - y_i| \longrightarrow \text{Not linear}$$

$$a, b \in \mathbb{R}$$

Add a variable e_i $\forall i \in \{1, 2, \dots n\}$

$$\left.\begin{aligned} e_i &\geq ax_i + b - y_i \\ e_i &\geq -(ax_i + b - y_i) \end{aligned}\right\} \Rightarrow e_i \geq |ax_i + b - y_i|$$

Final LP:

$$\text{minimize} \quad \sum_{i=1}^{n} e_i$$

$$e_i \geq ax_i + b - y_i$$

$$e_i \geq -(ax_i + b - y_i)$$

$$\text{minimize} \quad \sum_{i=1}^{n} e_i$$

$$e_i \geq a x_i + b - y_i$$
$$e_i \geq -(a x_i + b - y_i)$$

Variables: $a, b, e_1, \ldots, e_n$

A soln. to the constraints:

$a' \quad b' \quad e_1', \ldots e_n'$

$$e_1' \geq a' x_1 + b' - y_1$$
$$e_1' \geq -(a' x_1 + b' - y_1)$$

$$e_1' \geq |a' x_1 + b' - y_1|$$

Given particular values a', b':

the values for e_i that give minimum cost

are $= |a' x_i + b' - y_i|$ → equal to error

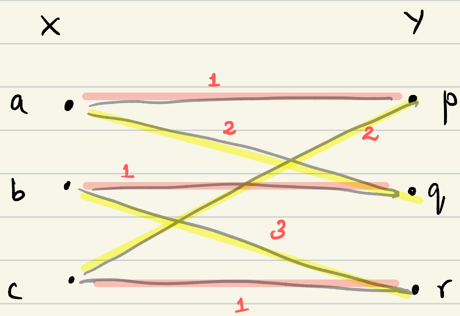
- Then the goal is to find values for a, b

that minimize this cost.

Example 4: Maximum-Weight matching in bipartite graphs

Given: a bipartite graph $G = (X, Y, E)$ s.t. $|X| = |Y|$

and $w: E \longmapsto \mathbb{N}$



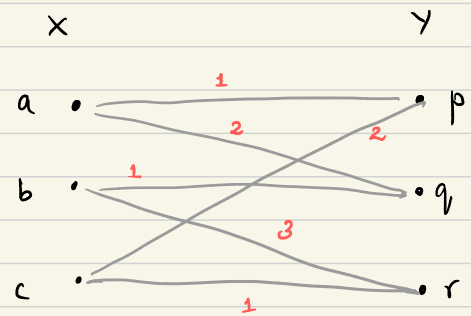
M_1:	M_2:
$a \overset{1}{-} p$	$a \overset{2}{-} q$
$b \overset{1}{-} q$	$b \overset{3}{-} r$
$c \overset{1}{-} r$	$c \overset{2}{-} p$
w: 3	7

Matching: a subset $M \subseteq E$ s.t.

for every $v \in X \cup Y$, exactly one edge in M is incident on v.

Maximum-Weight matching: Among all matchings, find the one with maximum weight

$$\sum_{e \in M} w_e$$



$x_{pa} + x_{pc} = 1$

Variables: x_e for each edge e.

$x_e = 0$ means $e \notin M$

$x_e = 1$ means $e \in M$

Constraints: for each vertex v:

$$\sum_{e \text{ is incident on } v} x_e = 1$$

$$x_e \in \{0, 1\}$$

Integer Linear Program:

$$\text{Maximize} \sum_{\forall e \in E} w_e x_e$$

subject to

$$\sum_{e \text{ is incident on } v} x_e = 1$$

$$0 \le x_e \le 1$$

$$\boxed{x_e \in \mathbb{Z}}$$ → Integrality constraint

LINEAR PROGRAM: General form

Variables $x_1, x_2, \dots x_n$ taking real values

Objective function

Maximize $c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

Subject to:

$$\left.\begin{array}{l} a_{11} x_1 + \dots + a_{1n} x_n \leq b_1 \\ a_{21} x_1 + \dots + a_{2n} x_n \leq b_2 \\ \vdots \\ a_{m1} x_1 + \dots + a_{mn} x_n \leq b_m \end{array}\right\}$$

Constraints

$c \in \mathbb{R}^n$ - a vector - seen as a column matrix $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

A: $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{m1} & & \dots & a_{mn} \end{bmatrix}$ $m \times n$ matrix

$b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ $m \times 1$

$$\boxed{\begin{array}{ll} \text{Maximize} & c^T x \\ \text{Subject to} & Ax \leq b \end{array}}$$

In this course:

- 1. Algorithms for solving LPs
- 2. Structural properties of LPs
- 3. Use of LPs in "combinatorial optimization"

$\downarrow$

These are discrete optimization problems,

- eg., maximum weight matching
  minimum vertex cover, etc.

EQUATIONAL FORM OF LPs:

$$\text{Maximize} \quad c^T x$$

$$\text{subject to:} \quad Ax = b$$

$$x \geq 0$$

$c \in \mathbb{R}^n$

$x \in \mathbb{R}^n$

$A: m \times n$

$b \in \mathbb{R}^m$

Example: convert following LP to equational form.

$$\text{Maximize} \quad 2x_1 + 3x_2 - 5x_3$$

$$\text{Subject to:} \quad \left.\begin{array}{l} x_1 - 4x_2 \leq 10 \\ x_1 + x_3 \leq 5 \\ x_1 \geq 0 \end{array}\right\}$$

$$\left.\begin{array}{l} x_1 - 4x_2 + S_1 = 10 \\ x_1 + x_3 + S_2 = 5 \\ S_1, S_2 \geq 0 \\ x_1 \geq 0 \end{array}\right\}$$

Not over yet since x_2, x_3 can take arbitrary values.

$$x_1 - 4x_2 + S_1 = 10$$
$$x_1 + x_3 + S_2 = 5$$

$$S_1 , S_2 \geq 0$$
$$x_1 \geq 0$$

$x_2 = x_2^+ - x_2^-$ $\qquad x_2^+ , x_2^- \geq 0$

$x_2 = 4$, $\quad x_2^+ = 4$
$\qquad x_2^- = 0$

$x_2 = -4$ $\quad x_2^+ = 0$
$\qquad x_2^- = 4$

Rewriting LP :

$$x_1 - 4(x_2^+ - x_2^-) + S_1 = 10$$
$$x_1 + 4(x_3^+ - x_3^-) + S_2 = 5$$

$$S_1 , S_2 \geq 0$$
$$x_1 \geq 0$$

$$x_2^+ , x_2^- , x_3^+ , x_3^- \geq 0$$

$$\text{Maximize} \quad 2x_1 + 3x_2 - 5x_3$$

$$x_1 - 4x_2 + S_1 = 10$$

$$x_1 + x_3 + S_2 = 5$$

values satisfying the constraints

$\langle x_1', x_2', x_3', s_1', s_2' \rangle$

$$S_1, S_2 \geq 0$$

$$x_1 \geq 0$$

$\downarrow$

$$\text{Maximize} \quad 2x_1 + 3(x_2^+ - x_2^-) - 5(x_3^+ - x_3^-)$$

$$x_1 - 4(x_2^+ - x_2^-) + S_1 = 10$$

$$x_1 + 4(x_3^+ - x_3^-) + S_2 = 5$$

x_1'

$x_2^+ = x_2'$ if $x_2' \geq 0$

$x_2^- = 0$

$x_2^+ = 0$ if $x_2' \leq 0$

$x_2^- = x_2'$

Similar for x_3^+, x_3^-, x_3'

s_1, s_2

$$S_1, S_2 \geq 0$$

$$x_1 \geq 0$$

$$x_2^+, x_2^-, x_3^+, x_3^- \geq 0$$

General form	Equational form
Maximize $2x_1 + 3x_2 - 5x_3$	Maximize $2x_1 + 3(x_2^+ - x_2^-) - 5(x_3^+ - x_3^-)$
Subject to: $x_1 - 4x_2 \leq 10$	$x_1 - 4(x_2^+ - x_2^-) + s_1 = 10$
$x_1 + x_3 \leq 5$	$x_1 + 4(x_3^+ - x_3^-) + s_2 = 5$
$x_1 \geqslant 0$	$s_1, s_2 \geqslant 0$
	$x_1 \geqslant 0$
	$x_2^+, x_2^-, x_3^+, x_3^- \geqslant 0$

For every feasible point in the left LP there is a feasible point for the right LP with the same cost and vice-versa

General form $\longrightarrow$ Equational form.

Note: Equational form can be seen as a general form.

$$a_{11} x_1 + \cdots + a_{1n} x_n = b_1$$

same as: $\sum_{j=1}^{n} a_{1j} x_j \leq b_1 \qquad - \sum_{j=1}^{n} a_{1j} x_j \leq -b_1$

Summary:

-1. Examples of problems modeled as LPs:

- Network flow
- Diet problem
- Fitting a line (Use of Modulus)
- Maximum-weight matching (Integer LP)

-2. LPs: General form (useful for geometric interpretations)

Equational form (useful for algebraic manipulations)