A geometric interpretation and an alternate proof of duality. For the rest of the questions assume we are given an LP maximize $Ax \leq b$, where $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and A is an $m \times n$ matrix. Further assume that the LP is non-degenerate and has an optimum at an extreme point x_0.

Some definitions.

- An *intersection point* $x \in \mathbb{R}^n$ is a point given by the intersection of some n hyperplanes $A_{i_1}x = b_{i_1}, \ldots, A_{i_n}x = b_{i_n}$ from A such that the rows $A_{i_1}, \ldots, A_{i_n}$ are linearly independent. We will denote these n rows corresponding to x as $A^x_1, \ldots, A^x_n$. We will write I for the set of all intersection points.
- Define F to be the set of all intersection points such that the cost function c can be written as a non-negative linear combination of the defining hyperplanes.

$$F = \{x \in I \mid c = \alpha_1 A^x_1 + \cdots + \alpha_n A^x_n \text{ for some non-negative } \alpha_1, \ldots, \alpha_n\}$$

- For an intersection point x we define an m-dimensional vector $y^x \in \mathbb{R}^m$ as follows. Suppose x is given by $A_{i_1}, \ldots, A_{i_n}$ and $c = \sum_{j=1}^{j=n} \alpha_j A_{i_j}$, then

$$\begin{aligned} y^x_{i_j} &= \alpha_j \quad \text{for all } j \in \{1, \ldots, n\} \\ y^x_l &= 0 \quad \text{for all } l \notin \{i_1, \ldots, i_n\} \end{aligned}$$

- Define $\overline{F} = \{y^x \mid x \in F\}$.

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1. What is the dual of the the LP: maximize $c^T x$ subject to $Ax \leq b$?

   Notice that the feasible points of the dual are m-dimensional vectors such that the cost is a non-negative combination of the m rows of A.

2. Answer true or false.

   - Every extreme point is an intersection point.
   - Every intersection point is an extreme point

3. Show that $\min\{c^T x \mid x \in F\} = c^T x_0$.

4. Show that $c^T x = b^T y^x$ for every $x \in F$.

5. Show that $\overline{F}$ is the set of extreme points of the dual.

6. Using the above observations, give a proof of the strong duality theorem.