1. Solve the following LPs using the simplex method

   (a)

$$\begin{array}{lrcrcrl} \text{Maximize} & x_1 & - & 2x_2 & + & x_3 & \\ \text{Subject to} & x_1 & + & 2x_2 & + & x_3 & \leq 12 \\ & 2x_1 & + & x_2 & - & x_3 & \leq 6 \\ & -x_1 & + & 3x_2 & & & \leq 9 \\ & x_1, & & x_2, & & x_3, & \geq 0 \end{array}$$

   (b)

$$\begin{array}{lrcrl} \text{Maximize} & 3x_1 & + & 5x_2 & \\ \text{Subject to} & x_1 & - & 2x_2 & \leq 6 \\ & x_1 & & & \leq 10 \\ & & & x_2 & \geq 1 \\ & x_1, & & x_2, & \geq 0 \end{array}$$

2. Provide an algorithm based on the simplex method to check if a given system of inequalities is feasible.

3. An LP in *general form* is given as: maximize $c^T x$ subject to $Ax \leq b$ where A is an $m \times n$ matrix, c and x are $n \times 1$, and b is $m \times 1$. Each constraint $A_i x \leq b_i$ is a half-space defined using the *hyperplane* $A_i x = b_i$. The general form LP is said to be *degenerate* if the feasible region contains a point obtained as an intersection of more than n hyperplanes.

   An LP in *equational form* is given as: maximize $c^T x$ subject to $Ax = b$, $x \geq 0$. Equational form LP is degenerate if there are several feasible bases corresponding to a single basic feasible solution.

   (a) Convert the following general form LP to equational form.

$$\begin{array}{lrcrcrl} \text{Maximize} & 5x_1 & - & x_2 & + & 2x_3 & \\ \text{Subject to} & x_1 & - & 6x_2 & + & x_3 & \leq 2 \\ & 5x_1 & + & 7x_2 & - & 2x_3 & \leq -4 \\ & 8x_1 & - & 10x_2 & + & 19x_3 & \leq 75 \\ & & & 5x_2 & + & 14x_3 & \leq 30 \end{array}$$

   (b) Convert the following equational form LP to general form.

$$\begin{array}{lrcrcrl} \text{Maximize} & 2x_1 & - & 3x_2 & + & 4x_3 & \\ \text{Subject to} & -x_1 & + & 2x_2 & - & x_3 & = 14 \\ & 5x_1 & - & 6x_2 & + & 12x_3 & = 20 \\ & x_1 & , & x_2 & , & x_3 & \geq 0 \end{array}$$

   (c) Show that if a general form LP is degenerate, then the corresponding equational form LP is also degenerate.

   (d) Show that if an equational form LP is degenerate, then the corresponding general form LP is also degenerate.

4. Suppose in an instance of LP, we have n variables that are unconstrained in sign. Show how they can be replaced by $n + 1$ variables that are constrained to be non-negative.

5. Write the duals for the following LPs:

- $$\begin{array}{llcccccl} \text{Maximize} & 2x_1 & - & 12x_2 & + & 20x_3 & & \\ \\ \text{Subject to} & 6x_1 & + & 9x_2 & + & 25x_3 & \leq 25 \\ & 2x_1 & - & 6x_2 & + & 3x_3 & = 15 \\ & 4x_1 & + & 7x_2 & - & 20x_3 & \geq 4 \\ & & & & & x_1 & \geq 0 \\ & & & & & x_2 & \leq 0 \\ & & & & & x_3 & \text{unrestricted} \end{array}$$

- $$\begin{array}{llcccccl} \text{Maximize} & 8x_1 & + & 3x_2 & - & 2x_3 & & \\ \\ \text{Subject to} & x_1 & - & 6x_2 & + & x_3 & \geq 2 \\ & 5x_1 & + & 7x_2 & - & 2x_3 & = -4 \\ & & & & & x_1 & \leq 0 \\ & & & & & x_2 & \geq 0 \end{array}$$

- $$\begin{array}{llcccccl} \text{Minimize} & -2x_1 & + & 3x_2 & + & 5x_3 & & \\ \\ \text{Subject to} & -2x_1 & + & x_2 & + & 3x_3 & \geq 5 \\ & 2x_1 & & & + & x_3 & \leq 4 \\ & & & 2x_2 & + & x_3 & = 4 \\ & & & & & x_1 & \leq 0 \\ & & & & & x_2 & \geq 0 \\ & & & & & x_3 & \text{unrestricted} \end{array}$$

6. Give an example of a primal-dual pair such that both are infeasible.

7. Take primal to be maximize $c^T x$ subject to $Ax = b$. The dual is then to minimize $b^T y$ subject to $A^T y = c$. Show that for every feasible solution $\overline{x}$ of primal and every feasible solution $\overline{y}$ of dual, we have $c^T \overline{x} = b^T \overline{y}$.