1. An oil refinery can buy two types of oil: light crude oil and heavy crude oil. The cost per barrel of these types of oil is respectively 11 and 9 dollars. The following quantities of gasoline, kerosene and jet fuel are produced per barrel of each type of oil.

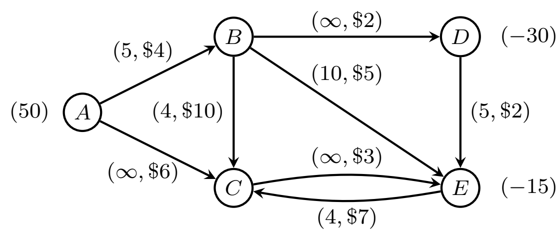
	Gasoline	Kerosene	Jet fuel
Light crude oil	0.4	0.2	0.35
Heavy crude oil	0.32	0.4	0.2

   From the above values, note that 5 percent of light crude oil and 8 percent of heavy crude oil are lost in the refining process. The refinery has contracted to deliver 1,000,000 barrels of gasoline, 400,000 barrels of kerosene and 250,000 barrels of jet fuel.

   Formulate a linear program for finding the number of barrels of each crude oil that satisfies the demand and minimizes the total cost (assuming fractional barrels are allowed).

2. Figure below shows a network-flow problem arising in the distribution of a single product from manufacturing plants (sources) to consumers (sinks). Node A is a source and nodes D and E are sinks. The number in paranthesis next to these nodes shows the product supplied or demanded at the particular node. Each directed edge is marked with two numbers - first coordinate is the maximum capacity of the route (edge) and the second coordinate is the cost per unit for transporting along that edge.

   Write an LP to determine the flow of the product along the network, satisfying the supply, demand and capacity constraints with minimum cost.



3. Convert the following LP to equational form:

$$
\begin{array}{lrcrcrl}
\text{Maximize} & 8x_1 & + & 3x_2 & - & 2x_3 & \\
\\
\text{Subject to} & x_1 & - & 6x_2 & + & x_3 & \geq 2 \\
 & 5x_1 & + & 7x_2 & - & 2x_3 & = -4 \\
 & & & & & x_1 & \leq 0 \\
 & & & & & x_2 & \geq 0
\end{array}
$$

4. Here is the set cover problem:

   Input. A universe D consisting of finite number of elements, and a family $S_1, S_2, \ldots, S_m$ of sets, with each $S_i \subseteq D$. Assume that $\bigcup_{i \in \{1,\ldots,m\}} S_i = D$.

   Goal. Find minimum size subset $W \subseteq \{1, \ldots, m\}$ such that $\bigcup_{i \in W} S_i = D$

   Write an ILP for the set cover problem.

5. (a) Write the ILP for finding min-cost perfect matching in a bipartite graph with edge weights. (Recall that a perfect matching is a subset of vertices M such that every vertex has exactly one edge from M incident on it.)

(b) Show that in the relaxed LP, every non-integral feasible point can be expressed as a convex combination of two distinct feasible points. (A convex combination of two vectors x and y is another point $\lambda x + (1 - \lambda)y$ for some $0 \leq \lambda \leq 1$)

6. Write an ILP for the shortest path problem.

   Shortest path: Given a directed graph with positive integer weights on edges, and two designated vertices s, t, find a minimum weight path from s to t.

7. Write an ILP for finding the maximum weight matching *with k edges* for a fixed k.

8. Consider a set of costs on the edges of the complete graph K_n with n nodes and let S be a given subset of vertices. We wish to find a subgraph of K_n that has an odd degree at nodes in S, an even (possibly zero) degree at all other nodes and as little total cost as possible. Write an ILP for this problem.

9. (a) Does Gaussian elimination preserve the dimension of the column space of the matrix?

   (b) Does Gaussian elimination preserve the column space?

10. Let x_1 and x_2 be solutions to $Ax = b$, $b \neq 0$. If $\alpha x_1 + \beta x_2$ is also a solution, what is the relation between α and β?

    Note: The relation derived above is called an *affine combination*.

11. As we have seen in the lecture, the set of solutions to $Ax = b$ form an affine subspace. Informally, an affine subspace is a translation of a subspace. Let S be an affine subspace of a vector space V. Show that, if S is a translation of a d-dimensional vector space, then there are $d + 1$ points in S that span whole of S. Here the span is considered as all *affine combinations* of the $d + 1$ points.

    An affine combination of $v_1, \ldots, v_d$ is a vector $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_d v_d$ such that $\Sigma_{i=1}^{i=d} \alpha_i = 1$. Notice that this is different from a convex combination.