LINEAR OPTIMIZATION

LECTURE 9

20/05/2021

The Simplex Method

An algorithm for solving LPs in equational form.

Idea:
- Start with a bfs
- deduce whether this bfs is optimum or if LP is unbounded.
- If not, move to a bfs with bigger cost.

Example 1:

$$\begin{aligned} \text{Maximize} \quad & x_1 + x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 1 \\ & x_1 \leq 3 \\ & x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

First bring it to equational form by adding slack variables x_3, x_4, x_5

$$\begin{aligned} -x_1 + x_2 + x_3 &= 1 \\ x_1 + x_4 &= 3 \\ x_2 + x_5 &= 2 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

$$A = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Notice that $B = \{3,4,5\}$ gives a bfs: $x_3 = 1$, $x_4 = 3$, $x_5 = 2$, $x_1 = x_2 = 0$

We will start with this bfs. First we will rearrange eqns. conveniently.

maximize $x_1 + x_2$

subj. to
$$\begin{aligned} -x_1 + x_2 + x_3 \quad\quad\quad &= 1 \\ x_1 \quad\quad\quad + x_4 \quad &= 3 \\ x_2 \quad\quad\quad + x_5 &= 2 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Initial bfs has x_3, x_4, x_5 as basic variables and x_1, x_2 as non-basic.

$$\boxed{\begin{aligned} x_3 &= 1 + x_1 - x_2 \\ x_4 &= 3 - x_1 \\ x_5 &= 2 - x_2 \\ \hline z &= x_1 + x_2 \end{aligned}}$$

Rewriting basic variables in terms of non-basic variables.

Writing the objective.

This representation is called a Tableau

The above tableau corresponds to a bfs: $\langle 0, 0, 1, 3, 2 \rangle$ with Cost : 0

Next step: Pivoting

If x_1 or x_2 is increased, the cost increases. Suppose we consider x_1.

How far can x_1 be increased?

$$
\begin{array}{rcl}
x_3 & = & 1 + x_1 - x_2 \\
x_4 & = & 3 - x_1 \\
x_5 & = & 2 - x_2 \\
\hline
z & = & x_1 + x_2
\end{array}
$$

Initial tableau T_0.

Next step: Pivoting

If x_1 or x_2 is increased, the cost increases. Suppose we consider x_1.

How far can x_1 be increased?

- The equation $x_3 = 1 + x_1 - x_2$ gives no restriction.
- $x_4 = 3 - x_1$ says $x_1 \leq 3$. Otherwise, $x_4 < 0$ and we don't get a feasible soln.

Let us therefore increase x_1 to 3.

- By doing this, x_4 becomes 0.
- We now move x_4 to RHS and x_1 to LHS.

$$
\boxed{\begin{array}{lllllll}
x_3 & = & 1 & + & x_1 & - & x_2 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 2 & & & - & x_2 \\
\hline
z & = & & & x_1 & + & x_2
\end{array}}
$$

Initial tableau T_0

$\downarrow$

Pivot step:

x_1 enters basis

x_4 leaves basis

$$
\begin{array}{lllllll}
x_3 & = & 1 & + & (3 - x_4) & - & x_2 \\
x_1 & = & 3 & - & x_4 & & \\
x_5 & = & 2 & & & - & x_2 \\
\hline
z & = & 3 & - & x_4 & + & x_2
\end{array}
$$

$\downarrow$

$$
\boxed{\begin{array}{lllllll}
x_3 & = & 4 & - & x_4 & - & x_2 \\
x_1 & = & 3 & - & x_4 & & \\
x_5 & = & 2 & & & - & x_2 \\
\hline
z & = & 3 & - & x_4 & + & x_2
\end{array}}
\qquad T_1
$$

Notice that T_1 is an equivalent system of equations to T_0.

By putting $x_4 = x_5 = 0$, we get a feasible soln. $\langle 3, 0, 4, 0, 2 \rangle$
In fact, this is a bfs with basis $\{1, 3, 5\}$. Cost $= 3$, larger than T_0.

$$
\begin{aligned}
x_3 &= 4 - x_4 - x_2 \\
x_1 &= 3 - x_4 \\
x_5 &= 2 - x_2 \\
\hline
z &= 3 - x_4 + x_2
\end{aligned}
$$

T_1

At T_1, look at z. Notice that increasing x_2 increases cost.

By how much can we increase x_2, by keeping $x_4 = 0$?

$x_3 = 4 - x_4 - x_2$ restricts $x_2 \leq 4$

$x_5 = 2 - x_2$ restricts $x_2 \leq 2$.

So we can go upto 2; as $x_2 \leq 2$ is stricter.

By setting $x_2 = 2$, we get $x_5 = 0$. We will bring x_5 to RHS.

$$
\begin{aligned}
x_3 &= 4 - x_4 - (2 - x_5) \\
x_1 &= 3 - x_4 \\
x_2 &= 2 - x_5 \\
\hline
z &= 3 - x_4 + (2 - x_5)
\end{aligned}
$$

$\downarrow$

$$
\begin{aligned}
x_3 &= 6 - x_4 + x_5 \\
x_1 &= 3 - x_4 \\
x_2 &= 2 - x_5 \\
\hline
z &= 5 - x_4 - x_5
\end{aligned}
$$

T_2

x_2 enters basis

x_5 leaves basis.

bfs: $\langle 3, 2, 6, 0, 0 \rangle$

$$
\begin{array}{lcl}
x_3 & = & 6 - x_4 + x_5 \\
x_1 & = & 3 - x_4 \\
x_2 & = & 2 \qquad\quad - x_5 \\
\hline
z & = & 5 - x_4 - x_5
\end{array}
\qquad T_2
$$

What next?

Once again, notice that T_2 is equivalent to the system that we started with. We have only rewritten the equations so that we have the same variables on RHS.

- T_2 corresponds to a bfs: $\langle 3, 2, 6, 0, 0 \rangle$
  Cost: 5

- Now, notice from $z = 5 - x_4 - x_5$ that increasing x_4 or x_5 will only decrease the cost.

- simplex method terminates here.

Claim: 5 is the optimum cost for this LP.

- For every feasible solution $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5)$ the cost is given by $\tilde{z} = 5 - \tilde{x}_4 - \tilde{x}_5$
  since $\tilde{x}_4, \tilde{x}_5 \geq 0$, $\tilde{z} \leq 5$.

- T_2 gives a feasible point, which is in fact a bfs, with cost $= 5$.

Today's goals:

- Exercises on simplex method to understand the algorithm.
- In later lectures, we will see a proof of correctness, and why each tableau corresponds to a bfs

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Example 2:

Maximize x_1

Subject to
$$\begin{aligned} x_1 - x_2 &\leq 1 \\ -x_1 + x_2 &\leq 2 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} x_1 - x_2 + x_3 \quad &= 1 \\ -x_1 + x_2 \quad + x_4 &= 2 \\ x_1,\ x_2,\ x_3,\ x_4 &\geq 0 \end{aligned}$$

$$x_1 = 1 - x_3 + x_2$$
$$x_4 = 2 + (1 - x_3 + x_2) - x_2$$

$$\boxed{\begin{aligned} x_3 &= 1 - x_1 + x_2 \\ x_4 &= 2 + x_1 - x_2 \\ \hline z &= x_1 \end{aligned}} \xrightarrow[x_3 \text{ leaves}]{x_1 \text{ enters}} \boxed{\begin{aligned} x_1 &= 1 - x_3 + x_2 \\ x_4 &= 3 - x_3 \\ \hline z &= 1 - x_3 + x_2 \end{aligned}}$$

bfs: $\langle 0, 0, 1, 2 \rangle$ bfs: $\langle 1, 0, 0, 3 \rangle$

cost: 0 cost: 1

$$\boxed{\begin{array}{rcl} x_1 &=& 1 - x_3 + x_2 \\ x_4 &=& 3 - x_3 \\ \hline z &=& 1 - x_3 + x_2 \end{array}}$$

bfs: $\langle 1, 0, 0, 3 \rangle$

Cost: 1

- Increasing x_2 will increase the cost.
- However, there is no constraint on how far x_2 can be increased.
  In every equation x_2 occurs with a non-negative coefficient.

In this case: Simplex terminates saying that the LP is unbounded.

- $x_2 = t \geq 0$ $\quad$ $x_1 = 1 - 0 + t$

  $x_3 = 0$ $\quad$ $x_4 = 3 - 0$

$$\begin{bmatrix} 1+t \\ t \\ 0 \\ 3 \end{bmatrix} \quad \text{for arbitrary } t \geq 0$$

is feasible, all coordinates are ≥ 0 and also satisfy the equations.

$\text{Cost} = 1 + t$

$\Rightarrow$ LP is unbounded.

Example 3:

Maximize $\quad -x_1 - x_2$

Subject to:

$$\begin{aligned} x_1 + 2x_2 &\leq 4 \\ x_2 &\leq 1 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_2 + x_4 &= 1 \\ x_1,\ x_2,\ x_3,\ x_4 &\geq 0 \end{aligned}$$

$$\boxed{\begin{array}{rcl} x_3 & = & 4 - x_1 - 2x_2 \\ x_4 & = & 1 - x_2 \\ \hline z & = & -x_1 - x_2 \end{array}}$$

bfs: $\langle 0, 0, 4, 1 \rangle$

Cost: 0

Example 4:

Maximize $-2x_1 + x_2$

Subj. to:
$$\begin{aligned} -x_1 + x_2 &\le 2 \\ 2x_1 + x_2 &\le 6 \\ x_1, \; x_2 &\ge 0 \end{aligned}$$

$$\begin{aligned} -x_1 + x_2 + x_3 \qquad &= 2 \\ 2x_1 + x_2 \qquad + x_4 &= 6 \\ x_1, \; x_2, \; x_3, \; x_4 &\ge 0 \end{aligned}$$

$$\boxed{\begin{aligned} x_3 &= 2 + x_1 - x_2 \\ x_4 &= 6 - 2x_1 - x_2 \\ \hline z &= -2x_1 + x_2 \end{aligned}}$$

x_2 enters

x_3 leaves

$$\boxed{\begin{aligned} x_2 &= 2 + x_1 - x_3 \\ x_4 &= 4 - 3x_1 + x_3 \\ \hline z &= 2 - x_1 - x_3 \end{aligned}}$$

– Optimum $= 2$ bfs: $\langle 0, 2, 0, 4 \rangle$

Example 5:

Maximize $\quad x_1 + x_2$

Subject to

$$
\begin{aligned}
-x_1 + x_2 &\leq 2 \\
x_2 &\leq 4 \\
x_1 + x_2 &\leq 9 \\
x_1 &\leq 6 \\
x_1 - x_2 &\leq 5 \\
x_1,\ x_2 &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= 2 + x_1 - x_2 \\
x_4 &= 4 - x_2 \\
x_5 &= 9 - x_1 - x_2 \\
x_6 &= 6 - x_1 \\
x_7 &= 5 - x_1 + x_2 \\
\hline
z &= + x_1 + x_2
\end{aligned}
$$

$$\xrightarrow[x_7 \downarrow]{x_1 \uparrow}$$

$$
\begin{aligned}
x_3 &= 7 - x_7 \\
x_4 &= 4 - x_2 \\
x_5 &= 4 + x_7 - 2x_2 \\
x_6 &= 1 + x_7 - x_2 \\
x_1 &= 5 - x_7 + x_2 \\
\hline
z &= 5 - x_7 + 2x_2
\end{aligned}
$$

$$x_2 \uparrow \ \big\downarrow \ x_6 \downarrow$$

$$
\begin{aligned}
x_3 &= 7 - x_7 \\
x_4 &= 3 - x_7 + x_6 \\
x_5 &= 2 - x_7 - 2x_6 \\
x_2 &= 1 + x_7 - x_6 \\
x_1 &= 6 - x_6 \\
\hline
z &= 7 + x_7 - 2x_6
\end{aligned}
$$

$$\xleftarrow[x_5 \downarrow]{x_7 \uparrow}$$

$$
\begin{aligned}
x_3 &= 5 + x_5 + 2x_6 \\
x_4 &= 1 + x_5 + 3x_6 \\
x_7 &= 2 - x_5 - 2x_6 \\
x_2 &= 3 - x_5 - 3x_6 \\
x_1 &= 6 - x_6 \\
\hline
z &= 9 - x_5 - 3x_6
\end{aligned}
$$

optimum. $\quad$ cost $= 9$

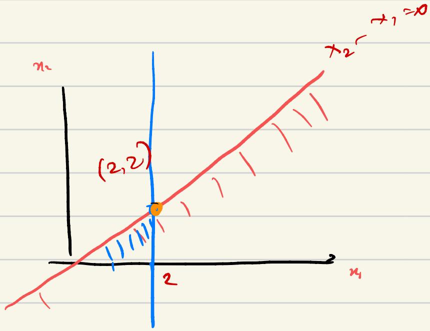
bfs : $\langle 6, 3, 5, 1, 0, 0, 2 \rangle$

Example 6:

maximize x_2

subject to:

$$-x_1 + x_2 \leq 0$$
$$x_1 \leq 2$$
$$x_1, x_2 \geq 0$$



$$A = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} x_3 &= 0 + x_1 - x_2 \\ x_4 &= 2 - x_1 \\ \hline z &= x_2 \end{aligned} \xrightarrow[x_3 \downarrow]{x_2 \uparrow} \begin{aligned} x_2 &= 0 + x_1 - x_3 \\ x_4 &= 2 - x_1 \\ \hline z &= x_1 - x_3 \end{aligned}$$

bfs: $\langle 0, 0, 0, 2 \rangle$ $\quad\quad$ $\langle 0, 0, 0, 2 \rangle$

$\{3, 4\}$ $\quad\quad$ $\{2, 4\}$

$\downarrow$ $x_1 \uparrow$, $x_4 \downarrow$

$$\begin{aligned} x_2 &= 2 - x_4 - x_3 \\ x_1 &= 2 - x_4 \\ \hline z &= 2 - x_4 - x_3 \end{aligned}$$

Optimum →

bfs: $\langle 2, 2, 0, 0 \rangle$

Cost : 2

Degeneracy in equational form:

An LP in equational form is said to be degenerate if several feasible bases correspond to a single bfs.

$\langle \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n \rangle$ B_1 B_2

- In such a case, at least one of the basic variables in the bfs has to be 0.

- We have seen an instance of this in Example 6.

Degenerate pivot step:

- A pivot step in which basis changes, but the cost does not change.

Cycling:

- a sequence of degenerate pivot steps that brings back to a previous tableau.

Later, we will see some pivoting rules that will prevent cycling.

Summary:

- Simplex method
- Detecting optimum, unboundedness — examples
- Degeneracy