LINEAR OPTIMIZATION

LECTURE 8

18/05/2021

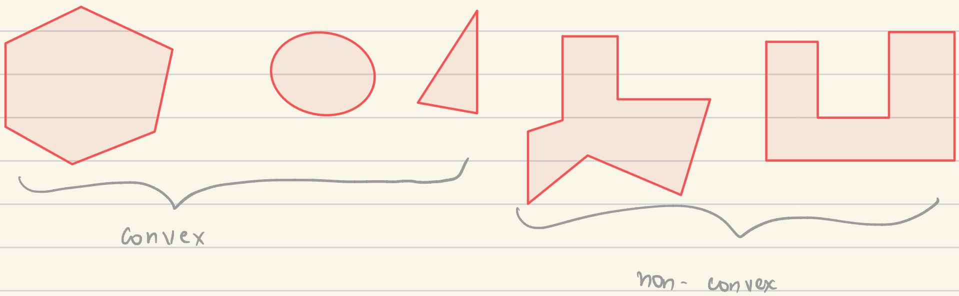
Bfs as vertices / extreme points of convex polyhedra

Convex combination: A convex combination of two points $x, y \in \mathbb{R}^n$ is another point in $\mathbb{R}^n$ given by:

$$\lambda x + (1-\lambda) y \quad \text{where} \quad 0 \leq \lambda \leq 1$$

Convex Set:

A set $S \subseteq \mathbb{R}^n$ is convex if for every pair of points $x, y \in S$ all its convex combinations $\lambda x + (1-\lambda) y$, $0 \leq \lambda \leq 1$, lie in S.



Lemma: If S_1 and S_2 are convex then $S_1 \cap S_2$ is convex.

Closed half-space: A closed half-space of $\mathbb{R}^n$ is the set of solutions to an inequality of the form:

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n \leq b_1$$

Remark: A closed half-space is convex.

Convex polyhedron:

A convex polyhedron is an intersection of finitely many closed half-spaces of $\mathbb{R}^n$.

A convex polyhedron is the set of solutions to finitely many inequalities of the form:

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m \end{aligned}}$$

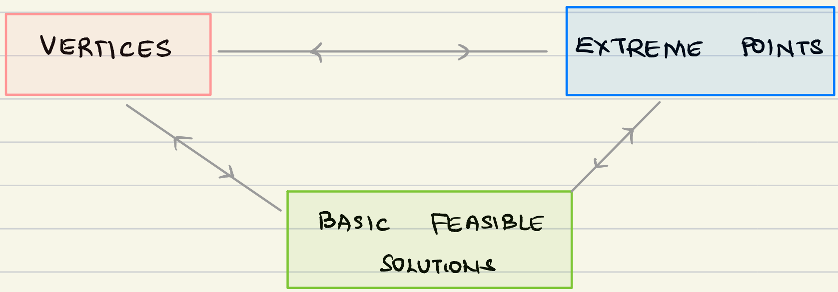
$$\downarrow$$

$$Ax \leq b$$

If a convex polyhedron is bounded, that is, it can be placed inside some large enough ball, it is called a convex polytope

$$a_{11}x_1 + a_{12}x_2 \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 \leq b_2$$
$$a_{31}x_1 + a_{32}x_2 \leq b_3$$

Goal for today's lecture:

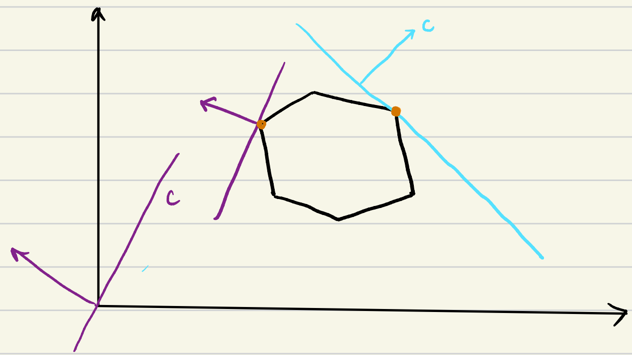


Notions of vertices, extreme points and bfs are same.

Vertex: A vertex of a convex polyhedron S is a point $x \in S$ such that:

there exists some non-zero cost vector $c \in \mathbb{R}^n$ s.t.

$c^T x > c^T y$ for all $y \in S$ with $y \neq x$.



Extreme point: A point x in a convex polyhedron S is an extreme point if

there are no two points $y, z \in S$ with y, z diffrent from x

s.t. x is a convex combination of y and z.

- Extreme points cannot be written as $\lambda y + (1-\lambda) z$ for $y, z \in S$ that are different from x.

- i.e., x does not lie in the segment yz.

Lemma: Each vertex is an extreme point.

Proof: Suppose v is a vertex.

There exists a $c \in \mathbb{R}^n$ s.t. $c^T v > c^T x$ for all $x \neq v$ in the polyhedron.

Suppose there exist $y, z \neq v$ in the polyhedron s.t.

$$v = \lambda y + (1-\lambda) z \quad \text{for some } 0 \leq \lambda \leq 1$$

$$c^T v = \lambda c^T y + (1-\lambda) c^T z \longrightarrow (1)$$

Now, we know that
$$c^T y < c^T v$$
$$c^T z < c^T v$$

$$\lambda c^T y + (1-\lambda) c^T z < \lambda c^T v + (1-\lambda) c^T v$$
$$< c^T v \longrightarrow (2)$$

(1) and (2) are inconsistent. This is a contradiction.

$\therefore$ There is no pair y, z in the polyhedron s.t. v is a combination of y, z.

Hence v is an extreme point.

Lemma: Each extreme point is a vertex.

To prove this lemma, we will make use of an alternate characterization of extreme points.

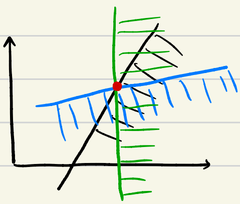
Proviso: We are given $Ax \leq b$

$A: m \times n$
$x: n \times 1$
$b: m \times 1$

- Each $A_i x = b_i$ is said to be the hyperplane defining the closed half-space $A_i x \leq b_i$

- We will assume that not more than n hyperplanes from $A_1 x = b_1, \ldots, A_m x = b_m$ pass through a point of $\mathbb{R}^n$

For example:



is not allowed since 3 hyperplanes pass through the red point. (degeneracy)

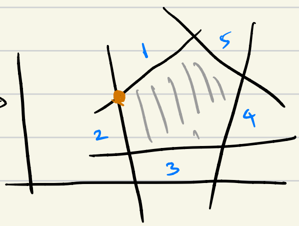
Theorem: A point in $Ax \leq b$ is an extreme point

iff

it can be expressed as an intersection of n linearly independent hyperplanes out of the hyperplanes defining $Ax \leq b$.

$a_{11}x_1 + a_{12}x_2 \leq b_1$
$a_{21}x_1 + a_{22}x_2 \leq b_2$
$\vdots$
$a_{51}x_1 + a_{52}x_2 \leq b_5$

$\rightarrow$



• satisfies:
$a_{11}x_1 + a_{12}x_2 = b_1$
$a_{21}x_1 + a_{22}x_2 = b_2$
$a_{31}x_1 + a_{32}x_2 < b_3$
$a_{41}x_1 + a_{42}x_2 < b_4$
$a_{51}x_1 + a_{52}x_2 < b_5$

Theorem: A point in $Ax \leq b$ is an extreme point

iff

it can be expressed as an intersection of n linearly independent hyperplanes out of the hyperplanes defining $Ax \leq b$.

Proof: ($\Leftarrow$): Suppose v is a point which is obtained as the intersection of n linearly independent hyperplanes out of $Ax \leq b$.

Separating out these hyperplanes, we can write:

$$A'v = b' \qquad A''v < b''$$

where A' has n linearly independent rows.

Due to our assumption about non-degeneracy we have a strict inequality for the second of constraints.

Notice that: A': $n \times n$ matrix and has rank n.

$\therefore$ v is the unique solution to $\left.\begin{matrix} \\ A'x = b' \end{matrix}\right\} \longrightarrow$ ①

To show: v is an extreme point.

Suppose v is not an extreme point.

$\therefore$ $v = \lambda u + (1-\lambda) w$ for u, w satisfying $Ax \leq b$.

$u, w \neq v$.

<u>To show:</u> v is an extreme point.

Suppose v is not an extreme point.

$\therefore$ $v = \lambda u + (1-\lambda) w$ for u, w satisfying $Ax \leq b$.
$u, w \neq v$.

Consider A':

$$A'u \leq b' \quad \text{and} \quad A'w \leq b'$$

$$b' = \begin{bmatrix} b'_1 \\ \vdots \\ b'_n \end{bmatrix}$$

Suppose there is an i s.t. $A'_i u < b'_i$

$$\lambda A'_i u + (1-\lambda) A'_i w < \lambda b'_i + (1-\lambda) A'_i w$$
$$< \lambda b'_i + (1-\lambda) b'_i$$
$$< b'_i$$

This is a contradiction to $v = \lambda u + (1-\lambda) w$

$$A'_i v = b'_i$$

whereas

$$\lambda A'_i u + (1-\lambda) A'_i w < b'_i$$

Therefore: $A'u = b'$ Hence $u = v$.

– This contradicts $v = \lambda u + (1-\lambda) w$ for $u, w \neq v$.

$\Rightarrow$ v is an extreme point.

$(\Rightarrow)$: Suppose v is an extreme point. We want to show that v lies in 'n' linearly independent hyperplanes.

We will show the following:

If v lies in the intersection of $< n$ linearly independent hyperplanes, then we can find $u, w \neq v$ s.t. u, w are feasible and
$$v = \lambda u + (1-\lambda) w.$$

i.e., If v lies in $< n$ hyperplanes, v is not an extreme point.

Consider a feasible point v which lies in $< n$ hyperplanes.

Let us write: $A'v = b'$ with A' having $< n$ linearly independent rows.
$A''v < b''$

Since A' has $< n$ linearly independent rows, it has $< n$ linearly independent columns.

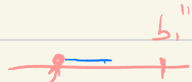
Hence recall that there is a non-trivial solution to
$$A'x = 0$$

Let this solution be x_0. So we have $A'x_0 = 0$.

i) $$A'(v + \epsilon x_0) = A'v + \epsilon A'x_0$$
$$= A'v$$
$$= b'$$

Adding any multiple of x_0 to v will still satisfy A' constraints

(ii) Now consider $A'' x \leq b''$

We have $A'' v < b''$ 

$$a''_{11} v_1 + \cdots + a''_{n1} v_1 < b''_1$$

$$\vdots$$

$$\underline{\qquad\qquad} < b''_{m-n}$$

$$A''(v + \epsilon x_0) = A''(v) + \epsilon A'' x_0$$

We can find an $\epsilon > 0$ s.t.

$$A''(v + \epsilon x_0) < b''$$

and $$A''(v - \epsilon x_0) < b''$$

From (i) and (ii) we get:

$v + \epsilon x_0$ and $v - \epsilon x_0$ are feasible.

$$\therefore \quad v = \tfrac{1}{2}(v + \epsilon x_0) + \tfrac{1}{2}(v - \epsilon x_0)$$

$\Rightarrow$ v is not an extreme point.

Lemma: Each extreme point is a vertex.

Proof: Let v be an extreme point.

We want a $c \in \mathbb{R}^n$. s.t. $c^T v > c^T x$ for every other feasible x.

From previous theorem: v lies in n linearly hyperplanes, written as:

$$A' v = b'$$
$$A'' v < b''$$

A' has n rows.

$$A' : \begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ & \vdots & & \\ a'_{n1} & a'_{n2} & \cdots & a'_{nn} \end{bmatrix} \begin{array}{l} \to A'_1 \\ \quad A'_i \\ \to A'_n \end{array} \qquad b' \begin{bmatrix} b'_1 \\ \vdots \\ b'_n \end{bmatrix}$$

Define $c^T = A'_1 + A'_2 + \cdots + A'_n$

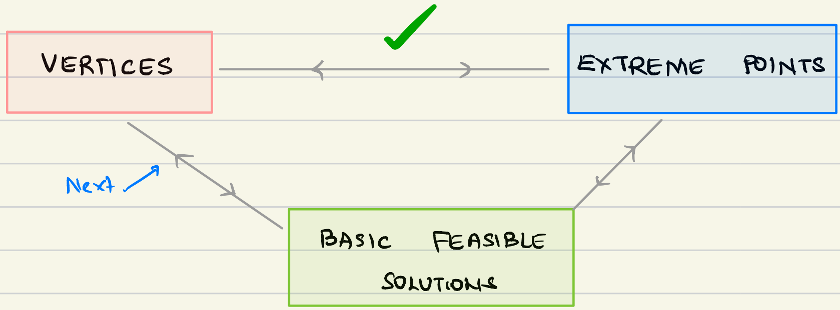
$$c^T v = A'_1 v + A'_2 v + \cdots + A'_n v = b'_1 + b'_2 + \cdots + b'_n$$

Take any other feasible x. $\exists i$ s.t. $A'_i v < b'_i$

$$\therefore c^T x = A'_1 x + A'_2 x + \cdots + A'_i x + \cdots + A'_n x$$

$$< b'_1 + \cdots + b'_n = c^T v$$

Goal for today's lecture:



Since we now move to basic feasible solutions, let us consider constraints in equational form.

$$Ax = b, \quad x \geq 0$$

Each $a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n = b_i$ is the intersection of two closed half spaces

$$a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \leq b_i$$

and $$a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \geq b_i$$

Therefore $Ax = b$, $x \geq 0$ gives a convex polyhedron.

Theorem: Let P be the convex polyhedron described by $Ax=0, x\geq 0$.
Let v be a point in P.

The following two conditions are equivalent:

(i) v is a vertex of P
(ii) v is a basic feasible solution of the LP.

Proof: (i) $\Rightarrow$ (ii). v is a vertex.
$\therefore \exists c$ s.t. $c^T v > c^T x$ for other $x \in P$.

We have seen that: for every $x \in P$, there exist a bfs $y \in P$
s.t. $c^T y \geq c^T x$

This is true for v too.
ie., $c^T y \geq c^T v$

But this is possible only if $v = y$,

$\therefore$ v is a bfs.

Theorem: Let P be the convex polyhedron described by $Ax = 0, x \geq 0$
Let v be a point in P.

The following two conditions are equivalent:

(i) v is a vertex of P
(ii) v is a basic feasible solution of the LP.

Proof of. (ii) $\Rightarrow$ (i):

Suppose v is a bfs.

there is a basis B for v.

Define c as follows: $c_j = 0$ if $j \in B$
$c_j = -1$ if $j \notin B$

(i) $c^T v = 0$, as v is a bfs with basis B
$x_j = 0$ for all $j \notin B$.

(ii) $c^T x$ for some $x \neq v$.,

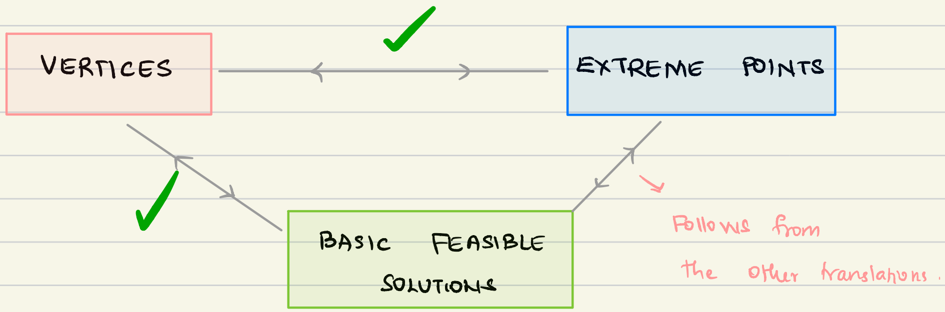
We have seen that $x_j > 0$ for some $j \notin B$.

$\therefore$ $c^T x < 0$. $\therefore$ $c^T x < c^T v$

This is true for every $x \neq v$.

$\Rightarrow$ v is a vertex.

Summary:



Reference: - Chapters 4.3 & 4.4 of the book:

Understanding and using Linear Programming
Matoušek & Gärtner

- Lectures 7, 8, 9 of Prof. Sundar Vishwanathan's course.