LINEAR OPTIMIZATION

LECTURE 7

13/05/2021

Problem Sheet 1: Solutions

1. Cost per barrel of light crude oil = \$11
heavy crude oil = \$9

	Gasoline	Kerosene	Jet fuel
Light oil	0.4	0.2	0.35
Heavy	0.32	0.4	0.2

Requirement: 1,000,000 barrels of gasoline
400,000 barrels of kerosene
250,000 barrels of jet fuel

LP to find no. of barrels of light and heavy oil that meets demand and minimizes cost.

Variables: l : no. of barrels of light oil (fractional barrels allowed)
h : no. of barrels of heavy oil

LP:

$$\begin{aligned} \text{minimize} \quad & 11l + 9h \\ \text{subject to:} \quad & 0.4l + 0.32h \geq 1000000 \\ & 0.2l + 0.4h \geq 400000 \\ & 0.35l + 0.2h \geq 250000 \\ & l, h \geq 0 \end{aligned}$$

2. <u>Variables:</u> for each edge ij a variable x_{ij}

x_{ij} denotes the no. of units of the product transported along edge $i \to j$

<u>LP:</u>

minimize $4x_{AB} + 6x_{AC} + 10x_{BC} + 2x_{BD} + 5x_{BE} + 3x_{CE} + 2x_{DE} + 7x_{EC}$

subject to:

$$x_{AB} + x_{AC} \leq 50$$ → supply constraint

$$x_{BD} - x_{DE} \overset{\geq}{=} 30$$
$$x_{BE} + x_{CE} + x_{DE} - x_{EC} \overset{\geq}{=} 15$$
→ demand constraint

$$0 \leq x_{AB} \leq 5$$
$$0 \leq x_{BC} \leq 4$$
$$\vdots$$
$$0 \leq x_{EC} \leq 4$$
→ capacity constraints

$$x_{AB} = x_{BC} + x_{BD} + x_{BE}$$
$$x_{AC} + x_{BC} + x_{EC} = x_{CE}$$
→ flow constraints

Assumption: – No node can store the product beyond its demand

3.

Step 1:

$$\begin{aligned} -x_1 + 6x_2 - x_3 + s_1 &= -2 \\ 5x_1 + 7x_2 - 2x_3 &= -4 \\ x_1 + s_2 &= 0 \\ x_2 &\geq 0 \end{aligned}$$

Step 2: Put $x_1 = y_1 - z_1$, $x_3 = y_3 - z_3$

$$\begin{aligned} -y_1 + z_1 + 6x_2 - x_3 + s_1 &= -2 \\ 5y_1 - 5z_1 + 7x_2 - 2x_3 &= -4 \\ y_1 - z_1 + s_2 &= 0 \end{aligned}$$

$$y_1, z_1, x_2, y_3, z_3 \geq 0$$
$$s_1, s_2 \geq 0$$

Objective function: $8y_1 - 8z_1 + 3x_2 - 2y_3 + 2z_3$

Alternately: Put $x_1' = -x_1$

maximize $-8x_1' + 3x_2 - 2y_3 + 2z_3$

subject to:

$$\begin{aligned} x_1' + 6x_2 - y_3 + z_3 + s_1 &= -2 \\ -5x_1' + 7x_2 - 2y_3 + 2z_3 &= -4 \end{aligned}$$

$$x_1', x_2, y_3, z_3 \geq 0$$
$$s_1 \geq 0$$

4. Set cover problem:

Variables: For each subset S_i, we associate a variable x_i which can take value 0 or 1.

ILP:

$$\boxed{\begin{array}{l} \text{Minimize} \quad \sum_{i=1}^{m} x_i \\ \text{Subject to:} \\ \qquad \sum_{e \in S_i} x_i \geq 1 \quad \text{for each } e \in D \\ \qquad x_i \in \{0,1\} \end{array}}$$

D

e_1: S_2, S_4, S_7 $\longrightarrow$ $x_2 + x_4 + x_7 \geq 1$

e_2 S_1, S_3 $\longrightarrow$ $x_1 + x_3 \geq 1$

$\vdots$ $\vdots$

e_n

5. a) ILP for min cost perfect matching in bipartite graphs with edge weights:

Input: Graph $G = (V, E, w)$ w: weight function.

Variables: x_e for each edge e.

ILP: minimize $\sum_{e \in E} w_e x_e$

subject to:

$$\sum_{e \text{ is incident on } v} x_e = 1 \quad \text{for every vertex } v \in V$$

$$0 \leq x_e \leq 1, \quad x_e \in \mathbb{Z} \quad \forall e$$

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b) Relaxed LP is obtained by removing the integrality constraint.

Let $\tilde{x}$ be a non-integral feasible solution.

- Suppose $\tilde{x}_{e_1}$ is non-integer. Let e_1 be between vertices a_1 and b_1

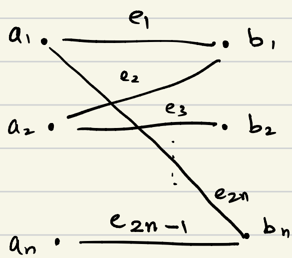
e_1: a_1 —— b_1

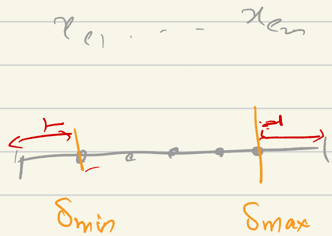
There exists an edge e_2: b_1 —— a_2 s.t. $\tilde{x}_{e_2}$ is non-integral due to the constraint $\sum_{e \text{ incident on } b_1} x_e = 1$

For the same reasons there exists edge:

$$e_3 : a_2 \text{ ——— } b_3 \quad \text{with } \tilde{x}_{e_3} \text{ non-integer.}$$

Since the graph is finite, continuing this process gives a cycle of edges for which $\tilde{x}$ is non-integer.





Let δ_{min} be min over $\tilde{x}_{e_i}$, $i = 1$, to $2n$

δ_{max} be max over $\tilde{x}_{e_i}$

We have $0 < \delta_{min} \leq \delta_{max} < 1$

Pick ϵ s.t. $0 < \epsilon < \min(\delta_{min}, 1 - \delta_{max})$

This value of ϵ ensures that $\tilde{x}_{e_i} + \epsilon$ and $\tilde{x}_{e_i} - \epsilon$ are in the interval $(0, 1)$ $\forall e_i$

Define two vectors $\tilde{x}_1$ and $\tilde{x}_2$ as follows:

$$\tilde{x}_1(e) = \begin{cases} \tilde{x}(e) & \text{if } e \notin \{e_1, \dots e_{2n}\} \\ \tilde{x}(e) + \epsilon & \text{if } e \in \{e_1, e_3, \dots e_{2n-1}\} \\ \tilde{x}(e) - \epsilon & \text{if } e \in \{e_2, e_4, \dots, e_{2n}\} \end{cases}$$

$$\tilde{x}_2(e) = \begin{cases} \tilde{x}(e) & \text{if } e \notin \{e_1, \dots, e_{2n}\} \\ \tilde{x}(e) - \epsilon & \text{if } e \in \{e_1, e_3, \dots, e_{2n-1}\} \\ \tilde{x}(e) + \epsilon & \text{if } e \in \{e_2, \dots, e_{2n}\} \end{cases}$$

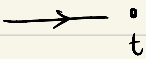
Notice that:

$$\tilde{x} = \frac{1}{2}\tilde{x}_1 + \frac{1}{2}\tilde{x}_2 .$$

Moreover: $\tilde{x}_1$ and $\tilde{x}_2$ are feasible solutions.

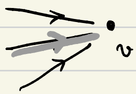
6. Constraints that ensure that a path from s to t is selected:

exactly one outgoing edge from s is selected
no incoming edge to s is selected

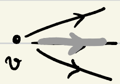


exactly one incoming edge to t
no outgoing edge from t

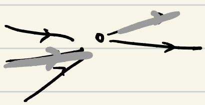
For every other vertex v:



at most 1 incoming edge

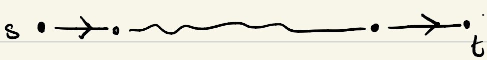


atmost 1 outgoing edge



no. of incoming = no. of outgoing

Claim: Every such selection gives a path from 's' to 't'.



ILP:

Minimize $\sum_{e \in E} w_e x_e$

Subject to: $\sum_{\substack{e \text{ outgoing} \\ \text{from 's'}}} x_e = 1$ $\qquad \sum_{\substack{e \text{ incoming} \\ \text{to } s}} x_e = 0$

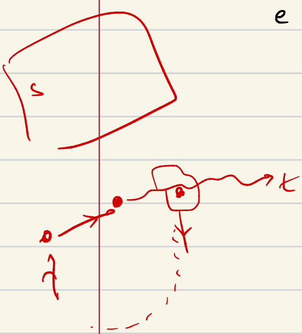
$$\sum_{\substack{e \text{ incoming} \\ \text{to 't'}}} = 1 \qquad \sum_{\substack{e \text{ outgoing} \\ \text{from } t}} x_e = 0$$

For all $v \notin \{s, t\}$

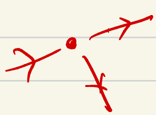
$$\sum_{\substack{e \text{ incoming} \\ \text{to } v}} x_e \leq 1$$

$$\sum_{\substack{e \text{ outgoing} \\ \text{from } v}} x_e \leq 1$$

$$\sum_{\substack{e \text{ incoming} \\ \text{to } v}} x_e = \sum_{\substack{e \text{ outgoing} \\ \text{from } v}} x_e .$$



$0 \leq x_e \leq 1$, $\quad x_e \in \mathbb{Z}$ for all e.



7. Maximum Weight matching with k-edges.

Input: Graph $G = (V, E, w)$ and number 'k'

↳ not necessarily bipartite.

ILP:

$$\text{maximize} \sum_{e \in E} w_e x_e$$

$$\text{subject to:} \quad \sum_{\substack{e \text{ incident} \\ \text{on } v}} x_e \leq 1 \quad \text{for all } v$$

$$\sum_{e \in E} x_e = k$$

$$x_e \in \{0, 1\}$$

8. Let vertices be $\{1, 2, \dots, n\}$

cost of edge $i - j$ be c_{ij} $(i < j)$

$S \subseteq \{1, 2, \dots, n\}$ is a given subset of vertices.

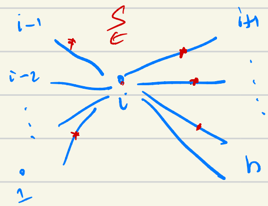
Variables: x_{ij} such that $i < j$

$x_{12} \quad x_{13} \quad \dots \quad x_{1n}$

$x_{23} \quad \dots \quad x_{2n}$

$\ddots$

$x_{n-1\,n}$



ILP: minimize $\sum c_{ij} x_{ij}$

subject to:

$$\forall i \in S: \quad \sum_{i<j} x_{ij} + \sum_{j<i} x_{ji} = 2s_i + 1$$

$$\forall i \notin S: \quad \sum_{i<j} x_{ij} + \sum_{j<i} x_{ji} = 2s_i$$

$$0 \leq x_{ij} \leq 1$$

$$s_i \geq 0 \quad \forall i$$

x_{ij}, s_i are all integers.

The ILP assigns a non-negative integer to each vertex i.

For vertices $i \in S$, the no. of edges selected should be $2s_i + 1$, making degree odd.

Similar for vertices $i \notin S$.

9. a) Yes. Gaussian elimination preserves the no. of linearly independent rows. Hence row rank is preserved. This implies that the column rank is also preserved.

b) No.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \xrightarrow{R_2 : R_2 - R_1} \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

A B

$$\text{Col space }(A) = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha + \beta \\ \alpha \\ \alpha \end{bmatrix}$$

$$\text{col-space }(B) = \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha + \beta \\ -\beta \\ \alpha \end{bmatrix}$$

Vector $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \in$ col-space (A)

but not in col-space (B)

Since
$$\alpha + \beta = 2$$
$$-\beta = 1$$
$$\alpha = 1$$
has no solution.

10. Given: $Ax_1 = b$ $Ax_2 = b$

$$A(\alpha x_1 + \beta x_2) = b$$

$$\alpha Ax_1 + \beta Ax_2 = b$$

$$\alpha b + \beta b = b$$

$$(\alpha + \beta) b = b$$

$$\Rightarrow \alpha + \beta = 1$$ – affine combination.

11. S: an affine subspace of a vector space V.

V: d-dimensional.

Let $v_1, v_2, \dots v_d$ be basis of V.

Let $S = u + V$

Hence every vector in S can be written as: $u + \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_d v_d$

$\alpha_1, \dots, \alpha_d \in \mathbb{R}$.

Claim: $u, \quad u+v_1, \quad u+v_2, \dots, u+v_d$ gives the required answer.

$\because \quad u + \alpha_1 v_1 + \dots + \alpha_d v_d$

$$= (1 - \alpha_1 - \alpha_2 \dots - \alpha_d)\, u + \alpha_1 (u + v_1) + \alpha_2 (u + v_2) + \dots + \alpha_d (u + v_d)$$

The above is an affine combination.

Solutions to Problem Sheet 2:

1. Basic feasible solutions of the following constraints:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 6 \\ 3 \end{bmatrix} \qquad m = 2.$$

Pairs of columns that are linearly independent.	Basic feasible soln.
$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $A^{(1)} \quad A^{(2)}$	$x_3 = 0 \quad x_4 = 0$ $x_1 + x_2 = 6$ $x_2 = 3$ $x_1 = 3, \quad x_2 = 3$ $[3 \ \ 3 \ \ 0 \ \ 0]$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A^{(1)} \quad A^{(4)}$	$x_2 = 0 \quad x_3 = 0$ $x_1 = 6, \quad x_4 = 3$ $[6, 0, 0, 3]$
$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $A^{(2)} \quad A^{(3)}$	$x_1 = 0 \quad x_4 = 0$ $x_2 + x_3 = 6$ $x_2 = 3$ $[0, 3, 3, 0]$
$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $A^{(2)} \quad A^{(4)}$	$[0 \ \ \cancel{6 \ \ -3} \ \ 0]$ not feasible.
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A^{(3)} \quad A^{(4)}$	$[0 \ \ 0 \ \ 6 \ \ 3]$

2. Given: x, y are feasible solns.

$$w = x - y \qquad K = \{ i \mid w(i) \neq 0 \}$$

We have:

$$x_1 A^1 + x_2 A^2 + \dots + x_n A^n = b$$
$$y_1 A^1 + y_2 A^2 + \dots + y_n A^n = b$$

Subtracting: $(x_1 - y_1) A^1 + (x_2 - y_2) A^2 + \dots + (x_n - y_n) A^n = 0$

i.e., $w_1 A^1 + w_2 A^2 + \dots + w_n A^n = 0$

Suppose $K = \{ i_1, i_2, \dots i_k \}$

We have: $w_{i_1} A^{i_1} + w_{i_2} A^{i_2} + \dots + w_{i_k} A^{i_k} = 0$

with $w_{i_j} \neq 0 \quad \forall j \in \{1, \dots k\}$

Hence $A^{i_1}, A^{i_2} \dots A^{i_k}$ are linearly dependent.

3. Suppose x is a bfs. Let its basis be B.

We have: $A_B x_B + A_N x_N = b$

But $x_N = 0$. Hence: $A_B x_B = b$

This has a unique soln. as all cols. of A_B are linearly independent

Any soln. y which satisfies $y(i) = 0$ iff $x(i) = 0$ will have

$$y_N = 0$$

$$\therefore \quad A_B y_B + A_N y_N = b$$

$$A_B y_B = b$$

$$\Rightarrow \quad x_B = y_B$$

As already shown: $x_N = y_N$. Hence $x = y$.

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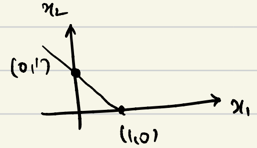
4. If x is not a bfs, here is a counter example.

Consider: $x_1 + x_2 = 1$

$x_1, x_2 \geq 0$



Let $\bar{x} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ $\bar{y} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

$x(i) = 0$ iff $y(i) = 0$. But $x \neq y$.