LINEAR OPTIMIZATION

LECTURE 6

11/05/2021

Basic Feasible Solutions

Today's lecture:

- 1. Given an LP in equational form, what are basic feasible solutions?

- 2. **Theorem:** If LP has an optimum, then there is a basic feasible solution that is optimal.

Proviso: Given LP in equational form:

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

$A: m \times n$

$n \geq m$

- the system of equations $Ax = b$ has at least one solution
- the rows of A are linearly independent.

Remark: From Proviso, row rank(A) = col rank(A) = m

$\downarrow$

dimension of column space = m

Notation: A : given matrix $\quad B \subseteq \{1, 2, \dots, n\}$ an index set

A_B: matrix consisting of only columns whose indices belong to B.

Example:

$$A = \begin{bmatrix} 5 & 4 & 3 & 10 & 2 \\ 10 & 6 & 7 & 21 & 0 \end{bmatrix}$$

$$B = \{1, 4\}$$

$$A_B = \begin{bmatrix} 5 & 10 \\ 10 & 21 \end{bmatrix}$$

Similarly for vectors: x and index set B,
x_B denotes the projection of x to coordinates indexed by B

$$x = [\,3 \quad 4 \quad 10 \quad 11 \quad 2\,]$$

$$B = \{3, 5\}$$

$$x_B = [\,10 \quad 2\,]$$

Basic Feasible Solutions:

A basic feasible solution of the LP

maximize $c^T x$ subject to $Ax = b$, $x \geq 0$ is

- a feasible solution $x \in \mathbb{R}^n$ for which
- there is an m-element index set $B \subseteq \{1, 2, \ldots, n\}$ s.t.
  - columns of A_B are linearly independent
  - $x_j = 0$ for all $j \notin B$

Example:

$$A = \begin{pmatrix} 1 & 5 & 3 & 4 & 6 \\ 0 & 1 & 3 & 5 & 6 \end{pmatrix} \qquad b = \begin{pmatrix} 14 \\ 7 \end{pmatrix}$$

$$x = (0 \;\; \underset{\uparrow}{2} \;\; 0 \;\; \underset{\uparrow}{1} \;\; 0)$$

$$B = \{2, 4\} \qquad A_B = \begin{pmatrix} 5 & 4 \\ 1 & 5 \end{pmatrix}$$

A characterization of basic feasible solutions:

A feasible soln. x is a bfs

iff

the columns of matrix A_K are linearly independent

where $K = \{ j \in \{1, 2, \ldots, n\} \mid x_j > 0 \}$

More notation:

- When B is fixed, we call the variables

  x_j with $j \in B$ as basic

  $j \notin B$ as non-basic

- An m-element set $B \subseteq \{1, 2, \dots, n\}$ such that

  columns of A_B are linearly independent

  will be called a basis.

Lemma: A feasible soln. x of an LP in equational form is basic

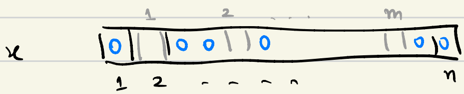
iff

the columne of matrix A_K are linearly independent where $K = \{ j \in \{1, 2, .., n\} \mid x_j > 0 \}$

Proof: $\Rightarrow$: Given a bfs x.

There exists an m-element set $B \subseteq \{1, 2, \dots, n\}$ s.t.

- columns of A_B are linearly independent
- $x_j = 0$ for all $j \notin B$



The set $K = \{ j \mid x_j > 0 \}$

Notice that $K \subseteq B$

Since columns of A_B are linearly independent; a subset of them is also linearly independent

This proves columns of A_K are lin. ind.

Lemma: A feasible soln. x of an LP in equational form is basic

iff

the columns of matrix A_K are linearly independent where $K = \{ j \in \{1, 2, .., n\} \mid x_j > 0 \}$

Proof: $\Leftarrow$:

We are given a feasible soln. x s.t. columns of A_K are linearly independent.

i) Can $|K| > m$? No, since dim(col-space of A) $= m$.

ii) $|K| = m$. We can take $B = K$.

iii) $|K| < m$. We have $|K|$ linearly independent columns.

$$A^{i_1} \quad A^{i_2} \quad \dots \quad A^{i_{|K|}}$$

There exists some column A^{j_1} which is outside the span of $A^{i_1} \dots A^{i_{|K|}}$

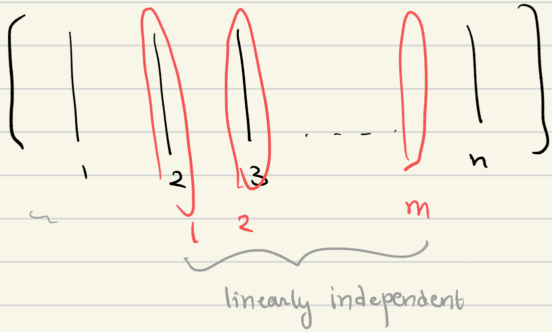
$\Rightarrow$ $A^{i_1} \; A^{i_2} \dots A^{i_{|K|}} \; A^{j_1}$ are linearly independent

If $|K| + 1 = m$, we are done. Otherwise continue the process till we get m linearly independent columns, using a similar argument.

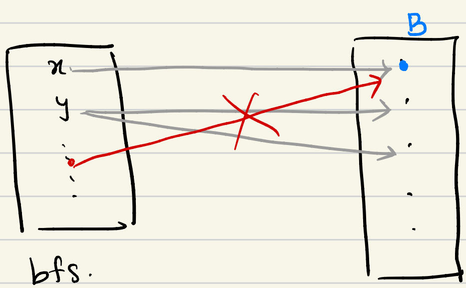
As long as there are less than m columns, there will be one outside this set which is not in its span.

Proposition: For every m-element set $B \subseteq \{1, 2, \dots, n\}$ that is a basis, there exists at most one feasible solution $x \in \mathbb{R}^n$ with $x_j = 0 \ \forall j \notin B$.

Illustration: $A =$



Suppose x is a basic f.s. Then there is a B



bfs.

m-element B s.t. cols. of A_B are lin. ind.

There cannot be two bfs

$\tilde{x}$ and $\tilde{\tilde{x}}$ s.t. both of them have the same set B as basis

Proposition: For every m- element set $B \subseteq \{1, 2, \dots, n\}$ that is a basis, there exists at most one feasible solution $x \in \mathbb{R}^n$ with $x_j = 0 \ \forall j \notin B$.

Proof: Given a B s.t. A_B is non-singular (cols. of A_B are lin. ind)

$$Ax = b$$

$$\begin{bmatrix} A^1 & A^2 & \cdots & A^n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b$$

$$x_1 A^1 + x_2 A^2 + \dots + x_n A^n = b \longrightarrow (*)$$

Group coordinates in B together.

$$B = \{i_1, i_2, \dots i_m\} \qquad N = \{1, 2, \dots n\} \setminus B$$
$$= \{j_1, j_2, \dots, j_{n-m}\}$$

Rewriting $(*)$:

$$x_{i_1} A^{i_1} + x_{i_2} A^{i_2} + \dots + x_{i_m} A^{i_m} + x_{j_1} A^{j_1} + \dots + x_{j_{n-m}} A^{j_{n-m}} = b$$

$$A_B x_B + A_N x_N = b$$

$$A_B x_B + A_N x_N = b$$

Suppose we have $x_j = 0$ for all $j \notin B$.

This means x_N is $\bar{0}$

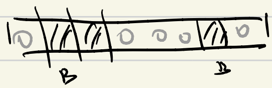
The above equation becomes: $A_B x_B = b$

— $A_B x_B = b$ has a unique solution.

If this unique solution is also non-negative, then we get a feasible solution of the LP.

Otherwise there is no feasible solution corresponding to B.

B — A_B is non-singular $\longrightarrow$ atmost 1 f.s.



f.s.

variables in N are 0.

$\rightarrow$ at most one such solution.

More notation:

- A basis $B \subseteq \{1, 2, \ldots, n\}$ will be called a feasible basis if the unique solution of the system $A_B x_B = b$ is non-negative.

Proposition: Assume that the cost function of the LP in equational form is bounded from above, that is
$c^T x \leq M$ for all feasible solutions x.

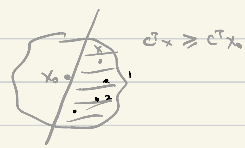
Then: for every feasible solution x_0, there exists a bfs $\tilde{x}$ with the same or larger cost, i.e.,

$$c^T \tilde{x} \geq c^T x_0 .$$

Proof: Consider a feasible solution x_0.

Define $S = \{ x \mid x \text{ is feasible and } c^T x \geq c^T x_0 \}$

S is non-empty since $x_0 \in S$



Let $\tilde{x}$ be a point in S with the maximum no. of zeroes. There could be several candidates, we choose one of them.

Claim: $\tilde{x}$ is a bfs.

$K = \{ j \mid \tilde{x}_j > 0 \}$. We want to show A_K is non-singular.

Suppose this is not the case: columns of A_K are linearly dependent.

There exist $|K|$ coefficients $v_1, v_2, \dots v_{|K|}$ s.t.

$$v_1 A_K^1 + v_2 A_K^2 + \dots + v_{|K|} A_K^{|K|} = 0$$

and some $v_i \neq 0$.

Suppose this is not the case: columns of A_K are linearly dependent.

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and some $v_i \neq 0$.

Construct an n-dimensional vector w:

$$w_j = \begin{cases} v_j & j \in K \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{array}{ccccccccc} & & 1 & & 2 & & k & & \\ \hline 0 & & v_1 & 0 & v_2 & \cdots & v_k & 0 & 0 \end{array}$$

Property 1: $Aw = 0$

$$w_1 A^1 + w_2 A^2 + \cdots + w_n A^n$$

$w_j = 0$ for all $j \notin K$.

The above sum: $v_1 A_K^1 + v_2 A_K^2 + \cdots + v_K A_K^{|K|}$

But this is zero as we have seen above.

Claim: We can assume:

- i) $c^T w \geq 0$
- ii) $w_j < 0$ for some coordinate j.

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- i) $c^T w \geq 0$
- ii) $w_j < 0$ for some coordinate j.

Proof: - Suppose $c^T w < 0$. We will negate all coordinates.

Doing this, we get $c^T w > 0$, and still $Aw = 0$

- If all coordinates of w are ≥ 0, we will get a contradiction.

Consider. $\tilde{x} + t \cdot w$

$$A(\tilde{x} + tw) = A\tilde{x} + tAw$$
$$= A\tilde{x} + 0$$
$$= b$$

If $\tilde{x} + tw$ additionally is non-negative, $\tilde{x} + tw$ is a feasible soln.

Now, we already know that $\tilde{x} \geq 0$.
So, if all coordinates of w are ≥ 0, then $\tilde{x} + tw \geq 0 \quad \forall t \geq 0$

$$c^T(\tilde{x} + tw) = c^T\tilde{x} + t \cdot c^T w$$

Since $c^T w > 0$, increasing t gives feasible solns. with arbitrarily increasing cost.
This contradicts hypothesis that cost is bounded.

Therefore some $w_j < 0$.

Suppose $c^T w = 0$

- If all coordinates are ≥ 0, we just negate w; $w' = -w$

This will give a vector w' s.t.

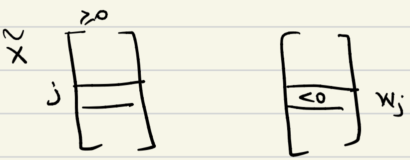
i) $c^T w' = 0$
ii) $w'_j < 0$ for at least one coordinate.

Claim: We can assume:

- i) $c^T w \geq 0$
- ii) $w_j < 0$ for some coordinate j.

Final argument: consider $\tilde{x} + tw$

- We already have $A(\tilde{x} + tw) = b$
- $\tilde{x} \geq 0$



In $\tilde{x} + tw$, by increasing t, the j^{th} coordinate tw will decrease.

While increasing 't', we will hit a value where

Final argument: consider $\tilde{x} + tw$

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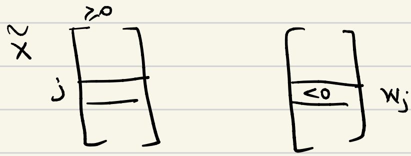


In $\tilde{x} + tw$, by increasing t, the j^{th} coordinate $\overset{tw}{}$ will decrease.

While increasing 't', we will hit a value where

$\tilde{x} + tw$ has more zeroes than $\tilde{x}$

Notice that for coordinates j s.t. $\tilde{x}_j = 0$, we also have $w_j = 0$

so $w_j < 0$ occurs at some $j \in K$.

$\Rightarrow$ increasing 't' will give us a feasible soln. with more zeroes.

$\rightarrow$ Let us say we get this at t^*.

$$c^T(\tilde{x} + t^* w) = c^T\tilde{x} + t^* \underbrace{c^T w}_{}$$

$$\geq c^T x_0 + 0$$

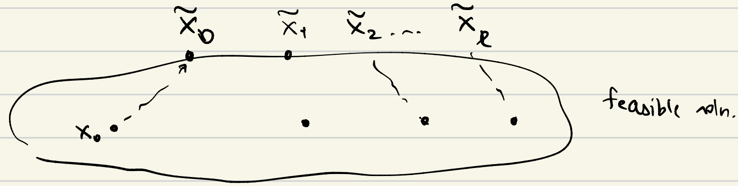
Hence $\tilde{x} + t^* w$ is a point in S with more zeroes than $\tilde{x}$. Contradiction. Hence columns of A_K are linearly independent.

Theorem: Consider an LP in equational form:

i) If there is at least one feasible solution and the objective function is bounded from above, then there exists an optimal solution.

ii) If there is an optimal solution, then there is a basic feasible solution that is optimal.

Illustration:



Notice that there are finitely many bfs.

- From previous proposition, for every feasible soln, there is a bfs with a better cost.

- Hence, maximum value is obtained from the maximum among bfs.

Naive algorithm to solve LP in equational form: (assuming cost is bounded from above)

- Enumerate all bfs:
  - choose a subset $B \subseteq \{1, 2, \dots, n\}$ of size m
  - check if columns of A_B are linearly independent
  - If yes: find the unique solution to $A_B x_B = b$
    if the solution ≥ 0 then we have a bfs by putting other coordinates to 0.

- Find a bfs with the maximum cost

Summary:

- Basic feasible solutions

- If LP is feasible and cost is bounded from above, an optimum exists at a bfs.

Reference: Chapter 4.2 of

Understanding and using Linear Programming
Matoušek & Gärtner