LINEAR OPTIMIZATION

LECTURE 5

06/05/2021

Goals: Understanding the set of solutions to $Ax = b$

$A: m \times n \qquad x: n \times 1 \qquad b: m \times 1$

To show:

$\boxed{\{x \mid Ax = b\} \text{ is an affine subspace of } \mathbb{R}^n}$ ✓

$\uparrow$

$\boxed{\{x \mid Ax = 0\} \text{ is a subspace of } \mathbb{R}^n}$ ✓

$\dim \{x \mid Ax = 0\} = n - k$ (k: no. of linearly independent columns of A)

$= n - r$ (r: no. of linearly independent rows of A)

Part 1: $\dim \{ x : Ax = 0 \} = n - k$ (k: no. of linearly independent columns of A)

Consider the system $Ax = 0$.

$$\boxed{\begin{aligned} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= 0 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= 0 \\ &\vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= 0 \end{aligned}}$$

Define $A^i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{im} \end{bmatrix}$ the i^{th} column of A

The system $Ax = 0$ can be rewritten as:

$$\boxed{x_1 A^1 + x_2 A^2 + \cdots + x_n A^n = \bar{0}}$$

Notice that $A^1, A^2, \ldots, A^n$ are in $\mathbb{R}^m$

The system $Ax = 0$ can be rewritten as:

$$\boxed{x_1 A^1 + x_2 A^2 + \dots + x_n A^n = \bar{0}}$$

Column space of A:

$A^1, A^2, \dots, A^n$ are the column vectors of A.

The subspace span $(A^1, A^2, \dots, A^n)$ is called column space of A.

Our hypothesis: dimension of column space $= k$

- There are k linearly independent columns in A.
  - $\hookrightarrow$ every other column can be written as a linear combination of these k columns.

Assume $A^1, A^2, \dots, A^k$ are linearly independent. Other columns can be written as a combination of these columns.

$$\boxed{\begin{aligned} A^{k+1} &= \beta_{1,k+1} A^1 + \beta_{2,k+1} A^2 + \dots + \beta_{k,k+1} A^k \\ &\vdots \\ A^n &= \beta_{1,n} A^1 + \beta_{2,n} A^2 + \dots + \beta_{k,n} A^k \end{aligned}}$$

$*$

Consider again the system $Ax=0$ which we view as:

$$x_1 A^1 + x_2 A^2 + \cdots + x_n A^n = 0 \quad \text{———} \quad \oplus$$

Recall that we are interested in showing that

$$\dim \{x : Ax=0\} = n-k.$$

Substituting $(*)$ in $\oplus$ gives:

$$\boxed{\begin{aligned} &(x_1 + \beta_{1,k+1}x_{k+1} + \beta_{1,k+2}\, x_{k+2} + \cdots + \beta_{1,n}\, x_n)\, A^1 \\ + &(x_2 + \beta_{2,k+1}\, x_{k+1} + \beta_{2,k+2}\, x_{k+2} + \cdots + \beta_{2,n}\, x_n)\, A^2 \\ + &\cdots \\ + &(x_k + \beta_{k,k+1}\, x_{k+1} + \beta_{k,k+2}\, x_{k+2} + \cdots + \beta_{k,n}\, x_n)\, A^k \\ &= 0 \end{aligned}} \quad (\maltese)$$

Since $A^1, \ldots, A^k$ are linearly independent, each of the coefficients:

$$x_i + \beta_{i,k+1}\, x_{k+1} + \cdots + \beta_{i,n}\, x_n = 0, \quad i=1,\ldots,k$$

$$
\begin{aligned}
&(x_1 + \beta_{1,k+1} x_{k+1} + \beta_{1,k+2} x_{k+2} + \dots + \beta_{1,n} x_n) A^1 \\
+\ &(x_2 + \beta_{2,k+1} x_{k+1} + \beta_{2,k+2} x_{k+2} + \dots + \beta_{2,n} x_n) A^2 \\
+\ &\dots \\
+\ &(x_k + \beta_{k,k+1} x_{k+1} + \beta_{k,k+2} x_{k+2} + \dots + \beta_{k,n} x_n) A^k \\
&= 0
\end{aligned}
\qquad (*)
$$

Since $A^1, \dots, A^k$ are linearly independent, each of the coefficients:

$$x_i + \beta_{i,k+1} x_{k+1} + \dots + \beta_{i,n} x_n = 0, \quad i = 1, \dots, k$$

---

* Therefore, every solution x to $Ax = 0$ satisfies:

$$
\begin{aligned}
x_1 &= -\beta_{1,k+1} x_{k+1} - \beta_{1,k+2} x_{k+2} \dots - \beta_{1,n} x_n \\
x_2 &= -\beta_{2,k+1} x_{k+1} - \beta_{2,k+2} x_{k+2} \dots - \beta_{2,n} x_n \\
&\vdots \\
x_k &= -\beta_{k,k+1} x_{k+1} - \beta_{k,k+2} x_{k+2} - \dots - \beta_{k,n} x_n
\end{aligned}
$$

* Similarly, every solution to the above system (in blue) is a solution to $Ax = 0$

x is a solution to $Ax=0$ iff

$$\boxed{\begin{aligned} x_1 &= -\beta_{1,k+1}\, x_{k+1} - \beta_{1,k+2}\, x_{k+2} \cdots - \beta_{1,n}\, x_n \\ x_2 &= -\beta_{2,k+1}\, x_{k+1} - \beta_{2,k+2}\, x_{k+2} \cdots - \beta_{2,n}\, x_n \\ &\ \ \vdots \\ x_k &= -\beta_{k,k+1}\, x_{k+1} - \beta_{k,k+2}\, x_{k+2} - \cdots - \beta_{k,n}\, x_n \end{aligned}}$$

- Notice that once we fix a value for $x_{k+1}, \dots, x_n$, the values for $x_1 \dots x_k$ get fixed.

- This indicates a dimension of $n-k$ for $\{x : Ax = 0\}$. We will now exhibit a basis of size $n-k$.

x is a solution to $Ax=0$ iff

$$\boxed{\begin{aligned} x_1 &= -\beta_{1,k+1}\,x_{k+1} - \beta_{1,k+2}\,x_{k+2} \cdots - \beta_{1,n}\,x_n \\ x_2 &= -\beta_{2,k+1}\,x_{k+1} - \beta_{2,k+2}\,x_{k+2} \cdots - \beta_{2,n}\,x_n \\ &\vdots \\ x_k &= -\beta_{k,k+1}\,x_{k+1} - \beta_{k,k+2}\,x_{k+2} - \cdots - \beta_{k,n}\,x_n \end{aligned}}$$

Define $n-k$ vectors: $U_{k+1}, U_{k+2}, \dots, U_n$ as follows:

- U_j, for $k+1 \leq j \leq n$, is obtained by setting $x_j = 1$ and the rest in $x_{k+1} \dots, x_n$ to 0.

$$U_{k+1} = \begin{bmatrix} -\beta_{1,k+1} \\ -\beta_{2,k+1} \\ \vdots \\ -\beta_{k,k+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad U_{k+2} = \begin{bmatrix} -\beta_{1,k+2} \\ -\beta_{2,k+2} \\ \vdots \\ -\beta_{k,k+2} \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad U_n = \begin{bmatrix} -\beta_{1,n} \\ -\beta_{2,n} \\ \vdots \\ -\beta_{k,n} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

<u>Claim 1</u>: $U_{k+1}, U_{k+2}, \dots, U_n$ are linearly independent.

<u>Claim 2</u>: $U_{k+1}, \dots, U_n \in \{x : Ax=0\}$

<u>Claim 3</u>: Every solution to $Ax=0$ can be written as a linear combination of $U_{k+1}, \dots, U_n$

$$U_{k+1} = \begin{bmatrix} -\beta_{1,k+1} \\ -\beta_{2,k+1} \\ \vdots \\ -\beta_{k,k+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad U_{k+2} = \begin{bmatrix} -\beta_{1,k+2} \\ -\beta_{2,k+2} \\ \vdots \\ -\beta_{k,k+2} \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad U_n = \begin{bmatrix} -\beta_{1,n} \\ -\beta_{2,n} \\ \vdots \\ -\beta_{k,n} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Claim 1: $U_{k+1}, \ldots, U_n$ are linearly independent.

Suppose $\alpha_{k+1} U_{k+1} + \cdots + \alpha_n U_n = 0$

$\alpha_{k+1} = 0$ due to the $k+1^{th}$ coordinate in U_{k+1}

$\vdots$

$\alpha_n = 0$ due to the n^{th} coordinate in U_n

$$U_{k+1} = \begin{bmatrix} -\beta_{1,k+1} \\ -\beta_{2,k+1} \\ \vdots \\ -\beta_{k,k+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad U_{k+2} = \begin{bmatrix} -\beta_{1,k+2} \\ -\beta_{2,k+2} \\ \vdots \\ -\beta_{k,k+2} \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad U_n = \begin{bmatrix} -\beta_{1,n} \\ -\beta_{2,n} \\ \vdots \\ -\beta_{k,n} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Claim 2: $AU_{k+1} = AU_{k+2} = \cdots = AU_n = 0$.

Recall:

x is a solution to $Ax = 0$ iff

$$\begin{aligned} x_1 &= -\beta_{1,k+1}x_{k+1} - \beta_{1,k+2}x_{k+2} \cdots - \beta_{1,n}x_n \\ x_2 &= -\beta_{2,k+1}x_{k+1} - \beta_{2,k+2}x_{k+2} \cdots - \beta_{2,n}x_n \\ &\vdots \\ x_k &= -\beta_{k,k+1}x_{k+1} - \beta_{k,k+2}x_{k+2} - \cdots - \beta_{k,n}x_n \end{aligned}$$

Notice that U_{k+1}, $U_{k+2}, \ldots U_n$ satisfy the above equations.

$$
v_{k+1} = \begin{bmatrix} -\beta_{1,k+1} \\ -\beta_{2,k+1} \\ \vdots \\ -\beta_{k,k+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad v_{k+2} = \begin{bmatrix} -\beta_{1,k+2} \\ -\beta_{2,k+2} \\ \vdots \\ -\beta_{k,k+2} \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \cdots \qquad v_n = \begin{bmatrix} -\beta_{1,n} \\ -\beta_{2,n} \\ \vdots \\ -\beta_{k,n} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}
$$

$$
\begin{aligned}
x_1 &= -\beta_{1,k+1} x_{k+1} - \beta_{1,k+2} x_{k+2} \cdots - \beta_{1,n} x_n \\
x_2 &= -\beta_{2,k+1} x_{k+1} - \beta_{2,k+2} x_{k+2} \cdots - \beta_{2,n} x_n \\
&\vdots \\
x_k &= -\beta_{k,k+1} x_{k+1} - \beta_{k,k+2} x_{k+2} - \cdots - \beta_{k,n} x_n
\end{aligned}
$$

$$
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{s.t.} \quad Ax = 0
$$

$$
\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \boxed{?}\, v_{k+1} + \boxed{?}\, v_{k+2} + \cdots \boxed{?}\, v
$$

$$U_{k+1} = \begin{bmatrix} -\beta_{1,k+1} \\ -\beta_{2,k+1} \\ \vdots \\ -\beta_{k,k+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad U_{k+2} = \begin{bmatrix} -\beta_{1,k+2} \\ -\beta_{2,k+2} \\ \vdots \\ -\beta_{k,k+2} \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad U_n = \begin{bmatrix} -\beta_{1,n} \\ -\beta_{2,n} \\ \vdots \\ -\beta_{k,n} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Claim 3: Every solution to $Ax=0$ can be written as a linear combination of $U_{k+1}, \dots, U_n$.

Let x be a solution to $Ax=0$. It satisfies:

$$x_1 = -\beta_{1,k+1} x_{k+1} - \beta_{1,k+2} x_{k+2} \cdots - \beta_{1,n} x_n$$

$$x_2 = -\beta_{2,k+1} x_{k+1} - \beta_{2,k+2} x_{k+2} \cdots - \beta_{2,n} x_n$$

$$\vdots$$

$$x_k = -\beta_{k,k+1} x_{k+1} - \beta_{k,k+2} x_{k+2} - \cdots - \beta_{k,n} x_n$$

Hence $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ x_{k+1} \\ \vdots \\ x_n \end{bmatrix} = x_{k+1} U_{k+1} + x_{k+2} U_{k+2} + \dots + x_n U_n$

Claims 1, 2, 3 show that $U_{k+1}, \ldots, U_n$ forms a basis for $\{x : Ax = 0\}$

- Hence $\dim(\{x : Ax = 0\}) = n - k$.

Summary of Part 1:

- $Ax = 0$ rewritten as $x_1 A^1 + x_2 A^2 + \cdots + x_n A^n = 0$
- $A^1, A^2, \ldots A^k$ are linearly independent. Write $A^{k+1}, \ldots, A^n$ in terms of $A^1, \ldots, A^k$.

$$\circledast \quad \boxed{\begin{aligned} A^{k+1} &= \beta_{1,k+1} A^1 + \beta_{2,k+1} A^2 + \cdots + \beta_{k,k+1} A^k \\ &\vdots \\ A^n &= \beta_{1,n} A^1 + \beta_{2,n} A^2 + \cdots + \beta_{k,n} A^k \end{aligned}}$$

- From this, show that x is a soln. to $Ax = 0$ iff.

$$\boxed{\begin{aligned} x_1 &= -\beta_{1,k+1} x_{k+1} - \beta_{1,k+2} x_{k+2} \cdots - \beta_{1,n} x_n \\ x_2 &= -\beta_{2,k+1} x_{k+1} - \beta_{2,k+2} x_{k+2} \cdots - \beta_{2,n} x_n \\ &\vdots \\ x_k &= -\beta_{k,k+1} x_{k+1} - \beta_{k,k+2} x_{k+2} - \cdots - \beta_{k,n} x_n \end{aligned}}$$

- Use this to exhibit a basis $U_{k+1}, \ldots, U_n$.

Next goal:

$$\boxed{\begin{aligned} \dim \{x \mid Ax = 0\} &= n - k \quad (k: \text{no. of linearly independent columns of } A) \\ &= n - r \quad (r: \text{no. of linearly independent rows of } A) \end{aligned}}$$

To show: $\dim \{x \mid Ax = 0\} = n - r$

A preprocessing step:

We have:

$$\boxed{\begin{aligned} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= 0 \\ &\vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= 0 \end{aligned}}$$

Define $A_1 = [a_{11} \quad a_{12} \quad a_{13} \quad \ldots \quad a_{1n}]^T$

$A_i = [a_{i1} \quad a_{i2} \quad \ldots \quad a_{in}]^T$

$\vdots$

$A_m = [a_{m1} \quad a_{m2} \quad \ldots \quad a_{mn}]^T$

Hypothesis: Rows $A_1, A_2, \ldots, A_r$ are linearly independent.
- Other rows $A_{r+1}, \ldots, A_n$ can be written as a combination of $A_1, \ldots, A_r$.

Removing a row that is a combination of other rows does not change the solution set.

Example:

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 + 4x_4 &= 0 \\ 3x_1 + 2x_2 + 7x_3 + x_4 &= 0 \\ 5x_1 + 5x_2 + 12x_3 + 5x_4 &= 0 \end{aligned}$$

Removing 3^{rd} row preserves solution set.

The pre processing step:

$$\begin{aligned} A_1 x &= 0 \\ A_2 x &= 0 \\ &\vdots \\ A_r x &= 0 \end{aligned}$$

linearly independent

$$\begin{aligned} \cancel{A_{r+1} x = 0} \\ \vdots \\ \cancel{A_m x = 0} \end{aligned}$$

Remove the equations $A_{r+1} x = 0, \dots, A_m x = 0$.

- Now we perform Gaussian elimination on the remaining rows to simplify further.

Gaussian elimination:

Involves repeated application of two operations:

- Exchange rows R_i and R_j
- $R_i \leftarrow R_i + \alpha \cdot R_j$ - add a multiple of some row to another row.

Both these operations preserve the set of solutions.

Consider our given set of r equations:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{r1} & a_{r2} & \cdots & a_{rn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- Gaussian elimination on $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \vdots & & \\ a_{r1} & a_{r2} & \cdots & a_{rn} \end{bmatrix}$

Gaussian elimination:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & a_{2n} \\ \vdots & & & \\ a_{r1} & a_{r2} & \cdots & a_{rn} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ 0 & \vdots & & \\ \vdots & & & \\ 0 & a'_{r2} & \cdots & a'_{rn} \end{bmatrix}$$

- Find the row from among 2 to r having the first non-zero coefficient when reading from left to right.

- Move this row to the second position.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & & a_{1n} \\ 0 & 0 & \cdots & a'_{2t_2} \cdots & a'_{2n} \\ 0 & \vdots & & \vdots & \\ 0 & \vdots & & & \\ \vdots & & & & \\ 0 & 0 & & a'_{rt_2} \cdots & a'_{rn} \end{bmatrix} \rightarrow$$

all 0's

Perform Gaussian elimination on the submatrix

(the other 0's will be unchanged)

In the end: the matrix looks like this:

$$\begin{bmatrix} a'_{1t_1} & \cdots & & & & & & & \\ 0 & 0 & 0 & a'_{2t_2} & \cdots & & & & \\ 0 & 0 & 0 & 0 & 0 \cdots & a'_{3t_3} & \cdots & & \\ & & \vdots & & & & & & \\ 0 & & \cdots & & & \cdots & 0 & a'_{rt_r} & \cdots \end{bmatrix}$$

with a'_{it_i} being the first non-zero coefficient in row i.

By exchanging columns, appropriately renaming the variables and scaling the rows,
we can assume the system to look like:

$$\begin{bmatrix} 1 & \cdots & & & & \\ 0 & 1 & \cdots & & & \\ 0 & 0 & 1 & & & \\ \vdots & & 0 & \ddots & & \\ 0 & 0 & 0 & 0 & 1 & \cdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Remark: Since we started with linearly independent rows, none of the rows will become fully $\bar{0}$ during the elimination.

Remark 2: x is a solution to $Ax=0$ iff it satisfies the modified system shown above.

- Once values for $x_{r+1}, \dots, x_n$ are fixed, the values of $x_1, \dots, x_r$ are determined.
- This suggests that the required dimension is $n-r$.
- We will now exhibit a basis to this end.

$$\begin{bmatrix} 1 & \cdots & & & & & \\ 0 & 1 & \cdots & & & & \\ 0 & 0 & 1 & & & & \\ \vdots & & 0 & \ddots & & & \\ 0 & 0 & 0 & 0 & 1 & \cdots & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Define: $$U_{r+1} = \begin{bmatrix} \alpha_{1,r+1} \\ \alpha_{2,r+1} \\ \vdots \\ \alpha_{r,r+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$ where $\alpha_{1,r+1} \dots \alpha_{r,r+1}$ are obtained from above

$\vdots$

$$U_n = \begin{bmatrix} \alpha_{1,n} \\ \vdots \\ \alpha_{r,n} \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$ where $\alpha_{1,n}, \dots \alpha_{r,n}$ are obtained from above system.

Claim 1: $U_{r+1}, \dots U_n$ are linearly independent

Claim 2: $AU_{r+1} = AU_{r+2} = \dots = AU_n = 0$

Claim 3: Every solution to $Ax = 0$ can be written as a linear combination of $U_{r+1}, \dots, U_n$.

$$\begin{bmatrix} 1 & \cdots & & & & \\ 0 & 1 & \cdots & & & \\ 0 & 0 & 1 & & & \\ \vdots & & 0 & \ddots & & \\ 0 & 0 & 0 & 0 & 1 & \cdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Define: $U_{r+1} = \begin{bmatrix} \alpha_{1,r+1} \\ \alpha_{2,r+1} \\ \vdots \\ \alpha_{r,r+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ where $\alpha_{1,r+1} \cdots \alpha_{r,r+1}$ are obtained from above

$\vdots$

$U_n = \begin{bmatrix} \alpha_{1,n} \\ \vdots \\ \alpha_{r,n} \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ where $\alpha_{1,n}, \dots \alpha_{r,n}$ are obtained from above system.

Claim 3: Every solution to $Ax=0$ is a combination of $U_{r+1} \dots U_n$.

Let $x = [x_1 \; x_2 \dots, x_n]^T$ be a solution to $Ax=0$.

- define $x' = x_{r+1} \cdot U_{r+1} + x_{r+2} U_{r+2} + \dots + x_n U_n$

  We will show that $x = x'$.

- Define $x'' = x - x'$. To show: $x'' = \bar{0}$

Observation 1: $x''_{r+1} = x''_{r+2} = \dots = x''_n = 0$

Observation 2: $Ax'' = Ax - Ax' = 0 - \sum_{i=r+1}^{n} x_i A U_i = 0$

Hence x'' is a solution to $Ax=0$.

$$\begin{bmatrix} 1 & - & - & - & - & - & - \\ 0 & 1 & - & - & - & - & \\ 0 & 0 & 1 & & & & \\ \vdots & & 0 & \ddots & & & \\ 0 & 0 & 0 & 0 & 1 & \cdots & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \longrightarrow \circledast$$

Define: $U_{r+1} = \begin{bmatrix} \alpha_{1,r+1} \\ \alpha_{2,r+1} \\ \vdots \\ \alpha_{r,r+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ where $\alpha_{1,r+1} \cdots \alpha_{r,r+1}$ are obtained from above

$\vdots$

$U_n = \begin{bmatrix} \alpha_{1,n} \\ \vdots \\ \alpha_{r,n} \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ where $\alpha_{1,n}, \ldots \alpha_{r,n}$ are obtained from above system.

Observation 1: $x''_{r+1} = x''_{r+2} = \cdots = x''_n = 0$

Observation 2: $Ax'' = Ax - Ax' = 0 - \sum_{i=r+1}^{n} x_j AU_i = 0$

Hence x'' is a solution to $Ax = 0$.

Since x'' is a solution to $Ax=0$, it satisfies the system $\circledast$ shown above.

Moreover $x''_{r+1} = \cdots = x''_n = 0$.

From $\circledast$ this means $x''_1 = x''_2 = \cdots = x''_r = 0$

This proves $x'' = x - x' = 0$

Hence the claim that x is a linear combination of $U_{r+1}, \ldots, U_n$

Claims 1, 2, 3 show that $v_{r+1}, \dots, v_n$ forms a basis to $\{x : Ax = 0\}$.

Hence $\dim \{x : Ax = 0\} = n - r$.

Summary of Part 2:

- Restrict $Ax = 0$ to the first r linearly independent rows.
- Perform Gaussian elimination to get a simplified system where

  $x_1, \dots x_r$ get determined by values of $x_{r+1} \dots x_n$.
- Exhibit a basis using this observation.

Summary:

$$\dim \{x \mid Ax = 0\} = n - k$$ (k: no. of linearly independent columns of A)

$$= n - r$$ (r: no. of linearly independent rows of A)

Corollary:

dimension of column space = dimension of row space of A

column rank = row rank

- This quantity is called rank of the matrix.