LINEAR OPTIMIZATION

LECTURE 4

04/05/21

Lecture 4: Equational form LP + Understanding $Ax = b$

Equational form of LP:

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$Ax = b$ → a set of equations

$x \geq 0$ → inequalities

Converting LP to equational form:

$$\begin{aligned} \text{maximize} \quad & 5x_1 + 4x_2 - 2x_3 \\ \text{subject to} \quad & 2x_1 + 3x_2 \leq 7 \\ & -3x_1 + 4x_2 + 5x_3 \geq 10 \\ & x_1 \geq 0 \\ & x_3 \leq 0 \end{aligned}$$

Step 1: Converting some of the constraints to $\leq$ form.

$$\begin{aligned} & 2x_1 + 3x_2 \leq 7 \\ & -3x_1 + 4x_2 + 5x_3 \geq 10 \quad \rightarrow \quad 3x_1 - 4x_2 - 5x_3 \leq -10 \\ & x_1 \geq 0 \quad \rightarrow \quad \text{No need to convert this} \\ & x_3 \leq 0 \end{aligned}$$

Step 2: Adding slack variables to get to equations

$$\begin{aligned} & 2x_1 + 3x_2 \leq 7 \\ & 3x_1 - 4x_2 - 5x_3 \leq -10 \\ & x_1 \geq 0 \\ & x_3 \leq 0 \end{aligned} \quad \rightarrow \quad \boxed{\begin{aligned} & 2x_1 + 3x_2 + s_1 = 7 \\ & 3x_1 - 4x_2 - 5x_3 + s_2 = -10 \\ & x_1 \geq 0 \\ & x_3 + s_3 = 0 \\ & s_1, s_2, s_3 \geq 0 \end{aligned}}$$

$$
\begin{aligned}
2x_1 + 3x_2 \quad + S_1 &= 7 \\
3x_1 - 4x_2 - 5x_3 \quad + S_2 &= -10 \\
x_1 &\geq 0 \\
x_3 \quad + S_3 &= 0 \\
S_1, S_2, S_3 &\geq 0
\end{aligned}
$$

Step 3: Non-negative variables

In the above LP, x_2 and x_3 are unconstrained. No explicit non-negativity constraints.

Replace:

$$x_2 = y_2 - z_2 \quad , \quad y_2, z_2 \geq 0$$

$$x_3 = y_3 - z_3 \quad , \quad y_3, z_3 \geq 0$$

$$
\boxed{
\begin{aligned}
2x_1 + 3x_2 \quad + S_1 &= 7 \\
3x_1 - 4x_2 - 5x_3 \quad + S_2 &= -10 \\
x_1 &\geq 0 \\
x_3 \quad + S_3 &= 0 \\
S_1, S_2, S_3 &\geq 0
\end{aligned}
}
\rightarrow
\begin{aligned}
2x_1 + 3y_2 - 3z_2 \quad + S_1 &= 7 \\
3x_1 - 4y_2 + 4z_2 - 5y_3 + 5z_3 + S_2 &= -10 \\
y_3 - z_3 \quad + S_3 &= 0 \\
x_1, y_2, z_2, y_3, z_3, S_1, S_2, S_3 &\geq 0
\end{aligned}
$$

Final LP:

$$
\boxed{
\begin{aligned}
\text{Maximize} \quad & 5x_1 + 4y_2 - 4z_2 - 2y_3 + 2z_3 \\
\text{Subject to:} \quad & 2x_1 + 3y_2 - 3z_2 \quad + S_1 = 7 \\
& 3x_1 - 4y_2 + 4z_2 - 5y_3 + 5z_3 + S_2 = -10 \\
& y_3 - z_3 + S_3 = 0 \\
& x_1, y_2, z_2, y_3, z_3, S_1, S_2, S_3 \geq 0
\end{aligned}
}
$$

Equational form of LP:

$$\boxed{\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}}$$

$Ax = b$ → a set of equations

$x \geq 0$ → inequalities

Equational form contains a set of equations $Ax = b$ along with simple inequalities $x \geq 0$ that are non-negativity constraints

Remark: $Ax = b$ (without the $x \geq 0$ constraints) can be solved efficiently using Gaussian elimination. The non-negativity constraints add substantial challenge to the problem, that require significantly different methods.

Next goal: Understanding the set of solutions to $Ax = b$

$A: m \times n \qquad x: n \times 1 \qquad b: m \times 1$

To show:

$\boxed{\{x \mid Ax = b\}$ is an affine subspace of $\mathbb{R}^n}$

↑

$\boxed{\{x \mid Ax = 0\}$ is a subspace of $\mathbb{R}^n}$

$\dim \{x \mid Ax = 0\} = n - k$ (k: no. of linearly independent columns of A)

$= n - r$ (r: no. of linearly independent rows of A)

Linear Algebra basics:

We will stick to real numbers.

Vector: an ordered tuple $(v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ of real numbers

$$\bar{v} = (v_1, v_2, \dots, v_n)$$

$$\bar{u} = (u_1, u_2, \dots, u_n)$$

$$\bar{v} + \bar{u} = (v_1 + u_1, v_2 + u_2, \dots, v_n + u_n)$$

$$t\bar{v} = (tv_1, tv_2, \dots, tv_n) \qquad \text{where } t \in \mathbb{R} \text{ (a scalar)}$$

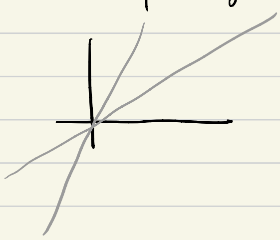
Linear subspace of $\mathbb{R}^n$: A set of vectors $V \subseteq \mathbb{R}^n$ s.t.

for all $\bar{u}, \bar{v} \in V$ and $t \in \mathbb{R}$, $\bar{u} + \bar{v} \in V$

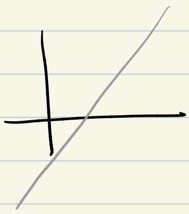
$t\bar{v} \in V$

Remark: A subspace contains the $\bar{0}$ vector.

Examples: Subspaces of $\mathbb{R}^2$ are $\{\bar{0}\}$, lines passing through origin, $\mathbb{R}^2$ itself



- line not passing through origin is not a subspace. ×



- Subspaces of $\mathbb{R}^3$ are $\{\bar{0}\}$, lines passing through origin, planes passing through origin, $\mathbb{R}^3$

$$\underline{ax_1 + bx_2 + cx_3} = 0$$

$\bar{u}$, $\bar{v}$ $t\bar{u}$, $\bar{u} + \bar{v}$

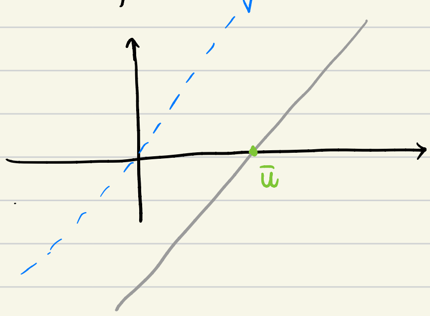
Affine subspace of $\mathbb{R}^n$: A set of the form:

$$\{ \bar{u} + \bar{v} \mid \bar{v} \in V \}$$

where V is a linear subspace of $\mathbb{R}^n$

- A subspace shifted by vector $\bar{u}$.

Examples: Affine subspaces of $\mathbb{R}^2$ are points, lines and $\mathbb{R}^2$



Affine subspace of $\mathbb{R}^3$ are points, lines, hyperplanes and $\mathbb{R}^3$.

Exercise: Show that $\{ x : Ax = 0 \}$ is a subspace of $\mathbb{R}^n$.

Proof:

- $\{ x \mid Ax = 0 \}$ is closed under addition.

Let $\bar{u}, \bar{v} \in \{ x \mid Ax = 0 \}$.

$\therefore \quad A\bar{u} = 0 \qquad A\bar{v} = 0$

$$A(\bar{u} + \bar{v}) = A\bar{u} + A\bar{v} = 0$$

$$\Rightarrow \quad \bar{u} + \bar{v} \in \{ x \mid Ax = 0 \}$$

- $\{ x \mid Ax = 0 \}$ is closed under scalar multiplication.

Let $\bar{u}$ s.t. $A\bar{u} = 0$.

$$\Rightarrow \quad A(t\bar{u}) = 0 \qquad \forall t \in \mathbb{R}$$

$$= t \cdot A\bar{u} = t \cdot \bar{0} = \bar{0}$$

Exercise: Assume $\{x : Ax = b\}$ is non-empty.

Show that $\{x : Ax = b\}$ is an affine subspace of $\mathbb{R}^n$.

Proof: Let $\bar{u}$ be a solution to $Ax=b$, i.e., $A\bar{u}=b$

Let $V = \{x : Ax=0\}$ V is a subspace.

Claim: $\{x : Ax=b\} = \bar{u} + V$

$\subseteq$: Pick $\bar{w}$ s.t. $A\bar{w}=b$

Write $\bar{w} = \bar{u} + \bar{w} - \bar{u}$

Now $A(\bar{w}-\bar{u}) = A\bar{w} - A\bar{u}$

$= b - b = 0$

$\Rightarrow \bar{w} - \bar{u} \in V$

$\Rightarrow \bar{w} \in \bar{u} + V$

$\supseteq$: Pick $\bar{u} + \bar{w} \in \bar{u} + V$

$A\bar{u} = b$ $A\bar{w} = 0$

$\therefore A(\bar{u}+\bar{w}) = b$

$\Rightarrow \bar{u} + \bar{w} \in \{x \mid Ax=b\}$

Recall goal: Understanding the set of solutions to $Ax = b$

$A: m \times n \quad x: n \times 1 \quad b: m \times 1$

To show:

$\{x \mid Ax = b\}$ is an affine subspace of $\mathbb{R}^n$ ✓

$\{x \mid Ax = 0\}$ is a subspace of $\mathbb{R}^n$ ✓

$\dim \{x \mid Ax = 0\} = n - k$ (k: no. of linearly independent columns of A)

$= n - r$ (r: no. of linearly independent rows of A)

Coming next:

- Linear independence
- dimension
- third observation above.

Linear dependence:

Vectors $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k$ are said to be linearly dependent

if $\exists$ reals $\alpha_1, \alpha_2, \ldots, \alpha_k$, not all of them 0

s.t. $\boxed{\alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2 + \ldots + \alpha_k \bar{u}_k = 0}$

Examples: 1) $\bar{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\bar{u}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$$2\bar{u}_1 - \bar{u}_2 = 0$$

2) $\bar{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\bar{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2 = 0$$

$$\alpha_1 + 2\alpha_2 = 0 \quad \text{— (1)}$$
$$2\alpha_1 + \alpha_2 = 0 \quad \text{— (2)}$$

From (1), $\alpha_1 = -2\alpha_2$

Substituting in (2): gives: $\alpha_2 = 0$

$$\Rightarrow \alpha_1 = 0$$

$\therefore$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are not linearly dependent.

Linear Independence:

$\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k$ are linearly independent if

for all reals $\alpha_1, \alpha_2, \ldots, \alpha_k$

$$\boxed{\alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2 + \ldots + \alpha_k \bar{u}_k = 0 \Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_k = 0}$$

Example: $\bar{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\bar{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\bar{u}_3 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$\alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2 + \alpha_3 \bar{u}_3 = 0$$

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$$
$$2\alpha_1 + \alpha_2 + 4\alpha_3 = 0$$

$\alpha_3 = -1$

$$\alpha_1 + 2\alpha_2 = 3$$
$$2\alpha_1 + \alpha_2 = 4$$

Solve for α_1, α_2.

- $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$ will be linearly dependent.

Basis of a linear subspace:

A set of vectors $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k$ forms a basis of a subspace V if:

- 1. they are linearly independent.
- 2. every vector $\bar{w} \in V$ can be written as a linear combination of $\bar{v}_1, \dots, \bar{v}_k$

$$\bar{w} = \beta_1 \bar{v}_1 + \beta_2 \bar{v}_2 + \dots + \beta_k \bar{v}_k$$

Theorem 1: Let $\{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_m\}$ and $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}$ be two bases of a subspace V. Then : $m = n$

Proof:

Idea: Assume $m < n$.

Let $S_0 = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_m\}$ $\quad T_0 = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}$

- We will construct $S_1 = S_0 \setminus \{\bar{u}_{i_1}\} \cup \{\bar{v}_1\}$ s.t. S_1 is also a basis
  - ↳ S_1 is obtained by removing some $\bar{u}_{i_1}$ and adding $\bar{v}_1$.
- We will repeat this construction to get:

$$S_2 = S_1 \setminus \{\bar{u}_{i_2}\} \cup \{\bar{v}_2\} \quad \text{s.t. } S_2 \text{ is a basis for } V$$

S_2 contains $\bar{v}_1, \bar{v}_2$ in the place of $\bar{u}_{i_1}, \bar{u}_{i_2}$

- By iterating this construction, we will get

$$S_m = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_m\} \text{ is a basis}$$

This contradicts the fact that $T = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_m, \bar{v}_{m+1}, \dots, \bar{v}_n\}$ is a basis.

In particular, it contradicts the hypothesis that vectors $\bar{v}_1, \dots, \bar{v}_n$ are linearly independent.

- This will then prove that $m = n$.

Before we proceed, we make use of an intermediate definition:

Span: $\text{span}(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_k) = \{ \sum_{i=1}^{k} \alpha_i \bar{u}_i \mid \alpha_i \in \mathbb{R} \}$

$\hookrightarrow$ all linear combinations of $\bar{u}_1, \dots, \bar{u}_k$

Lemma: Suppose $S = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_m\}$ and $\bar{v} = \sum_{i=1}^{m} \alpha_i u_i$ s.t. $\alpha_1 \neq 0$.

Let $S' = \{\bar{v}, \bar{u}_2, \bar{u}_3, \dots, \bar{u}_m\}$

Then $\text{span}(S) = \text{span}(S')$

Proof:

$$\bar{v} = \alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2 + \dots + \alpha_m \bar{u}_m \longrightarrow \circledast$$

$\text{span}(S) \subseteq \text{span}(S')$:

Pick $\bar{w} \in \text{span}(S)$; $\bar{w} = \beta_1 \bar{u}_1 + \beta_2 \bar{u}_2 + \dots + \beta_m \bar{u}_m$

$$\bar{w} = \beta_1 \left(\frac{1}{\alpha_1} \bar{v} - \frac{\alpha_2}{\alpha_1} \bar{u}_2 - \dots - \frac{\alpha_m}{\alpha_1} \bar{u}_m \right) + \beta_2 \bar{u}_2 + \dots + \beta_m \bar{u}_m$$

$$\bar{w} = \beta_1' \bar{v} + \beta_2' \bar{u}_2 + \dots + \beta_m' \bar{u}_m \quad \text{for appropriate } \beta_i'.$$

$\Rightarrow \bar{w} \in \text{span}(S')$

$\text{span}(S') \subseteq \text{span}(S)$:

Pick $\bar{w} \in \text{span}(S')$; $\bar{w} = \beta_1 \bar{v} + \beta_2 \bar{u}_2 + \dots + \beta_m \bar{u}_m$

$$= \beta_1 (\alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2 + \dots + \alpha_m \bar{u}_m) + \dots + \beta_m \bar{u}_m$$

$$= \beta_1' \bar{u}_1 + \dots + \beta_m' \bar{u}_m$$

Hence $\bar{w} \in \text{span}(S)$

Back to proof of Theorem 1:

$$S_0 = \{ \bar{u}_1, \bar{u}_2, \dots, \bar{u}_m \} \qquad T = \{ \bar{v}_1, \bar{v}_2, \dots, \bar{v}_m \}.$$

Since S_0 is a basis: $\bar{v}_1 = \alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2 + \dots + \alpha_m \bar{u}_m$

Assume $\alpha_1 \neq 0$.

We will now show that $\{ \bar{v}_1, \bar{u}_2, \dots, \bar{u}_m \}$ is a basis

— i) $\{ \bar{v}_1, \bar{u}_2, \dots \bar{u}_m \}$ are linearly independent

Suppose $\beta_1 \bar{v}_1 + \beta_2 \bar{u}_2 + \dots + \beta_m \bar{u}_m = 0$

Writing $\bar{v}_1$ in terms of $\bar{u}_1 \dots \bar{u}_m$:

$$\beta_1 (\alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2 + \dots + \alpha_m \bar{u}_m) + \beta_2 \bar{u}_2 + \dots + \beta_m \bar{u}_m = 0$$

$$(\beta_1 \alpha_1) \bar{u}_1 + (\beta_1 \alpha_2 + \beta_2) \bar{u}_2 + \dots + (\beta_1 \alpha_1 + \beta_m) \bar{u}_m = 0$$

Since $\bar{u}_1, \dots, \bar{u}_m$ are linearly independent, we have:

$$\beta_1 \alpha_1 = 0$$
$$\beta_1 \alpha_2 + \beta_2 = 0$$
$$\vdots$$
$$\beta_1 \alpha_m + \beta_m = 0$$

But we know that $\alpha_1 \neq 0$. Hence $\beta_1 = 0$

This also implies that $\beta_2 = \beta_3 = \dots = \beta_m = 0$

Hence $\{ \bar{v}_1, \bar{u}_2, \dots, \bar{u}_m \}$ are linearly independent.

- ii) $\text{span}(\{ \bar{v}_1, \bar{u}_2, \dots, \bar{u}_m \}) = \text{span}(\{ \bar{u}_1, \bar{u}_2, \dots, \bar{u}_m \})$

from previous lemma.

At the i^{th} iteration of this process, we have:

$S_i = \{ \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_i, \bar{u}_{i+1}, \bar{u}_{i+2}, \ldots, \bar{u}_m \}$ (through appropriate renumbering)

as a basis.

- As S_i is a basis, $\bar{v}_{i+1}$ can be written as a combination of vectors in S_i.

i.e., $\bar{v}_{i+1} = \beta_1 \bar{v}_1 + \beta_2 \bar{v}_2 + \ldots + \beta_i \bar{v}_i + \alpha_{i+1} \bar{u}_{i+1} + \ldots + \alpha_m \bar{u}_m$

Atleast one of the $\alpha_i \neq 0$, as otherwise it contradicts $v_1, v_2 \ldots v_{i+1}$ being linearly independent.

- Assume $\alpha_{i+1} \neq 0$.

Using arguments as before, we can show that:

$S_{i+1} = \{ \bar{v}_1, \bar{v}_2, \ldots \bar{v}_i, \bar{v}_{i+1}, \bar{u}_{i+2}, \ldots, \bar{u}_m \}$ is a basis

- Continuing this process, we get $S_m = \{ \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m \}$ to be a basis.

This is a contradiction to $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m, \bar{v}_{m+1}, \ldots \bar{v}_n$ being linearly independent.

Dimension of a subspace: cardinality of its basis

From previous theorem, all bases have the same cardinality, which we call the dimension

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Recall goal: Understanding the set of solutions to $Ax = b$

$A: m \times n \qquad x: n \times 1 \qquad b: m \times 1$

To show:

$\boxed{\{x \mid Ax = b\} \text{ is an affine subspace of } \mathbb{R}^n}$ ✓

↑

$\boxed{\{x \mid Ax = 0\} \text{ is a subspace of } \mathbb{R}^n}$ ✓

$\dim \{x \mid Ax = 0\} = n - k$ (k: no. of linearly independent columns of A)

$= n - r$ (r: no. of linearly independent rows of A)

↓

Next lecture