LINEAR OPTIMIZATION

LECTURE 21

PRIMAL-DUAL ALGORITHM FOR MINIMUM VERTEX COVER

Input: an undirected graph $G = (V, E)$ with weights on vertices $w : V \longrightarrow \mathbb{N}$

Output: A vertex cover of minimum weight

A vertex cover is a subset $V' \subseteq V$ s.t. every edge has one of its endpoints in V'.

- This problem is a computationally hard problem (corresponding decision version is NP-hard)

- In lecture 3, we have modeled this problem as an ILP, and we considered its LP relaxation.

  From the optimum of the LP relaxation, we constructed an integral soln., ie, a vertex cover. We showed that the vertex cover that we obtain this way is at most twice the size of the optimum vertex cover.

- We proved the above in the unweighted setting. However similar argument works in the weighted setting too.

Today's goal: A primal-dual algorithm for min vertex cover that gives a 2-approximate solution.

References:
- Lecture 7 of Approximation Algorithms course by Debmalya Panigrahi
- Lecture notes of Advanced Algorithms course by Shuchi Chawla.

Part 0: Approximation factors from primal-dual algorithms

- The problem at hand is modeled as an ILP.
- Then we consider an LP relaxation

  Assume this is the primal and it is a minimization problem.

* We are now going to consider primal-dual algorithms that do not necessarily return the optimum of primal. Instead the algorithm stops at some feasible soln. of dual (D), which we call Algo optimum (D). From this soln., a primal feasible soln. is generated. We call this Algo optimum (P).

- The situation is pictorially represented below.

- The distance between Algo optimum (P) and ILP optimum (P) is denoted as Approx. ratio.

  Notice that this distance is smaller than the distance between the (P) and (D) optima given by the algo. Therefore the Algo ratio gives a bound for the Approx. ratio.

- For the primal-dual algo, we will reason about the Algo ratio.

Part 1: Primal and dual for vertex cover problem

ILP:

$$\min \sum_{u \in V} w_u x_u$$

subject to:

$$x_u + x_v \geq 1 \quad \forall e = (u,v) \in E$$

$$0 \leq x_u \leq 1$$

$$x_u \text{ integral}$$

Relaxed LP: We will remove both the integrality constraint and the $x_u \leq 1$ constraint.

Notice that the LP obtained thus will have an optimum smaller than ILP optimum.

Primal:

$$\min \sum_{u \in V} w_u x_u$$

subj. to

$$x_u + x_v \geq 1 \quad \forall e = (u,v) \in E$$

$$x_u \geq 0 \quad \forall u \in V$$

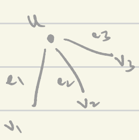
Dual:

$$\max \sum_{e \in E} y_e$$

subj. to

$$\sum_{e \text{ incident on } u} y_e \leq w_u \quad \forall u \in V.$$

$$y_e \geq 0$$



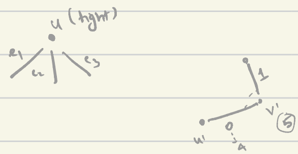
Part 2: The primal-dual algorithm

- Initially set all dual variables to 0.

- Iterative step: Suppose we have a dual soln. y.

  Let $T \subseteq V$ be the set of vertices s.t. the dual constraint is tight for y:

$$\sum_{e \text{ incident on } u} y_e = w_u$$



  - Increase the value of some edge $e = (u, v)$ s.t. neither u nor v are tight ($u \notin T$, $v \notin T$) until some vertex becomes tight.

- Termination: When above iterative step cannot be done, that is, when for every edge (u, v) at least one of its vertices is tight.

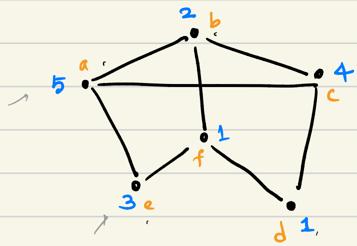
From the final dual solution y obtained, there is a natural primal soln x:

$$x_u = 1 \quad \text{if dual constraint for } u \text{ is tight for } y$$
$$= 0 \quad \text{otherwise}$$

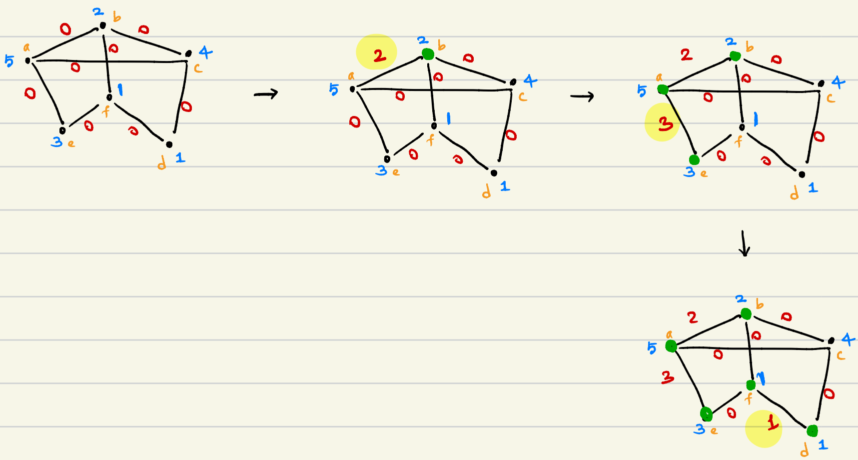
- From the termination condition, this 'x' gives a vertex cover.

- We will prove that this vertex cover has weight at most twice the weight of optimal vertex cover. Before proving this, we illustrate the primal-dual algo on an example.

Example:



f, a, c



Termination

Vertex cover given by algo: $\{ b, a, e, d, f \}$

- Cost: $2 + 5 + 3 + 1 + 1 = 12$

Optimal vertex cover = $\{ f, a, c \}$, with cost: 10

Notice that $12 \leq 2 \cdot 10$

Part 3: Proof of 2- approximation.

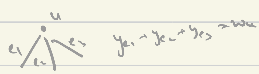
Recall the picture:

We will show that : Algo optimum (P) $\leq$ 2. Algo optimum (D).

This will show that Algo optimum (P) $\leq$ 2. ILP optimum.

- Consider the primal soln. x and dual soln. y obtained at the end of the primal- dual algorithm.

Let $T = \{ u \mid x_u = 1 \}$



Cost of primal $= \sum_{u \in T} w_u$

$$= \sum_{u \in T} \Big(\sum_{\substack{e \text{ incident} \\ \text{on } u}} y_e \Big)$$

each edge e gets counted at most twice

$$\leq 2 \sum_{e \in E} y_e$$

$$= 2. \text{ Cost of dual}$$

Summary:

- How to employ primal-dual method to get approximate algorithms.
- A primal-dual algorithm for min. vertex cover that gives a 2-approximation.