LINEAR OPTIMIZATION

LECTURE 2

GOALS

- 1. Solving an LP: intuition in 2D
- 2. Integer linear programs (ILPs)
- 3. LP relaxations of ILPs
- 4. Maximum weight matching

References: Chapters 1 & 3 of

Understanding and Using Linear Programming

Jiří Matoušek, Bernd Gärtner

Part 1:

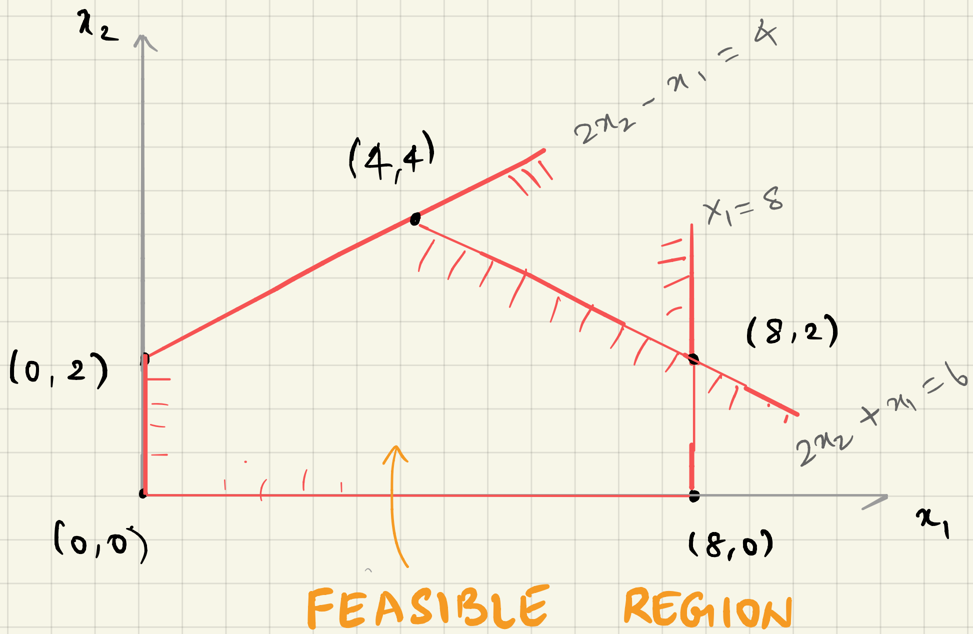
Solving an LP: intuition in 2D

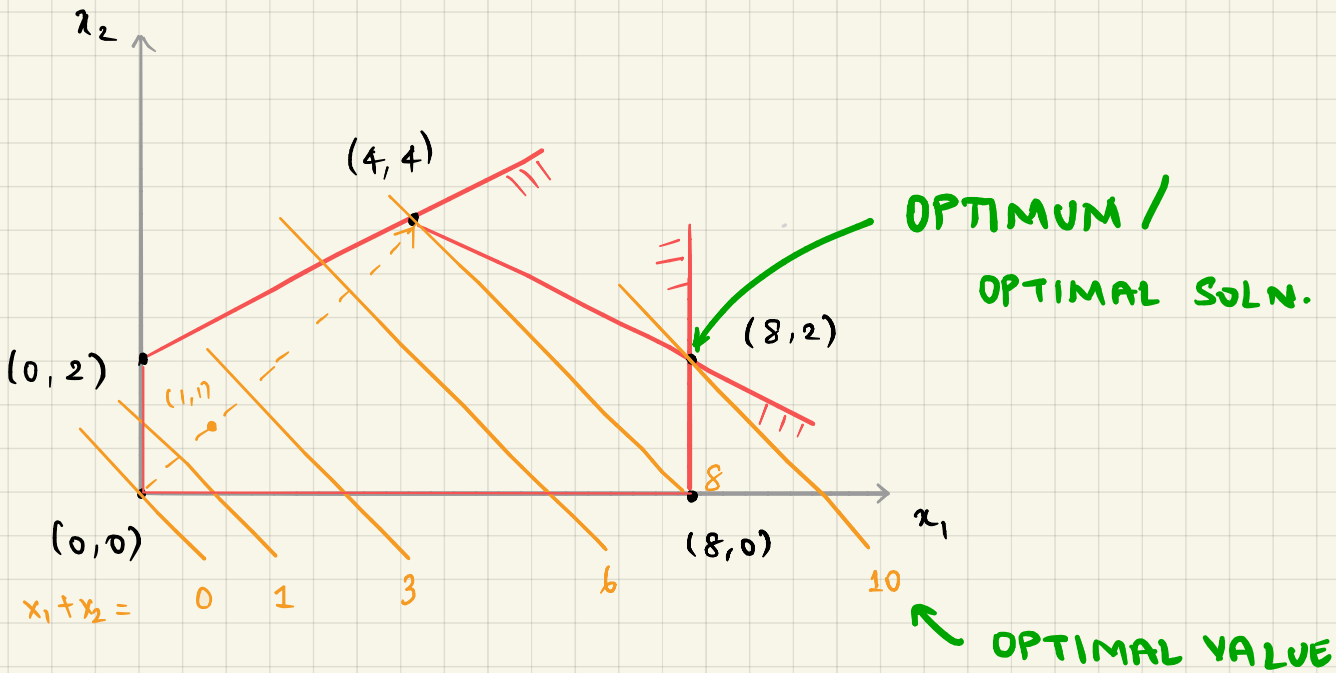
Maximize $\quad x_1 + x_2$

subject to

$$x_1 \geq 0$$
$$x_2 \geq 0$$

$$2x_2 - x_1 \leq 4$$
$$x_1 \leq 8$$
$$2x_2 + x_1 \leq 6$$



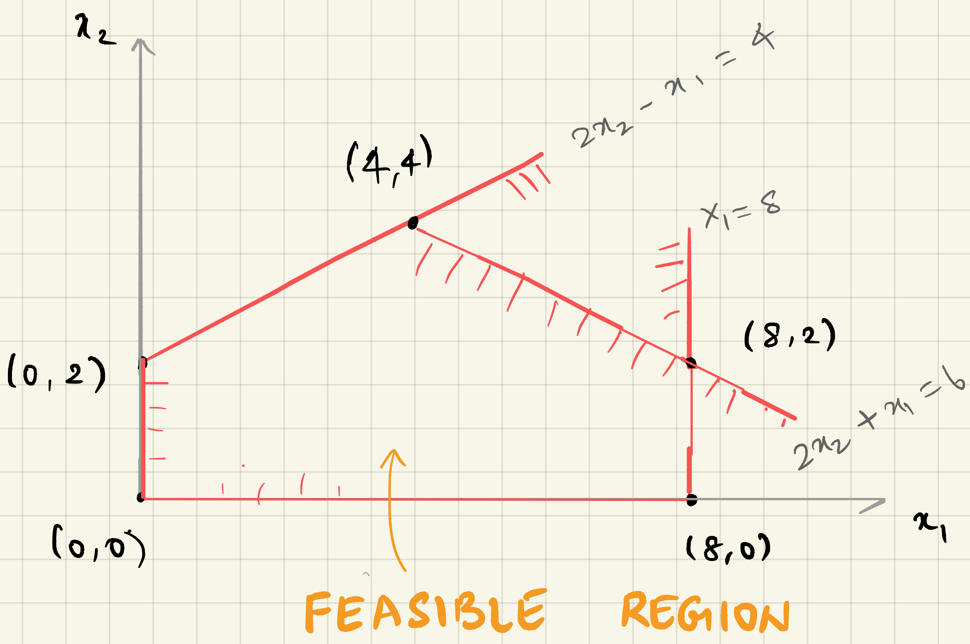


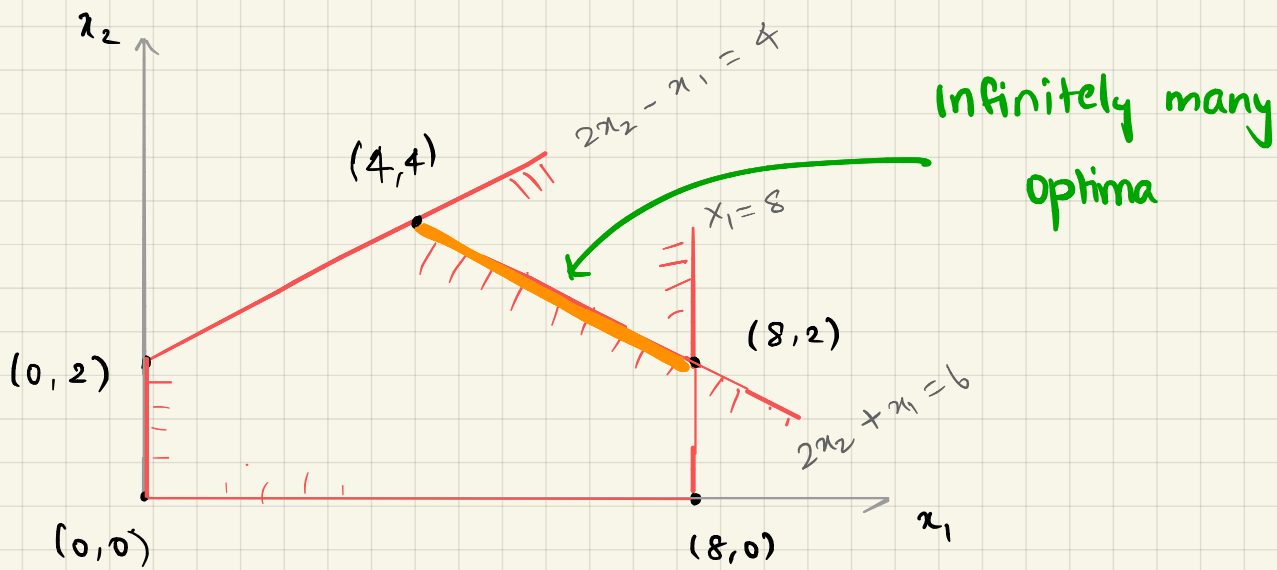
Maximize $\quad 2x_2 + x_1$ $\quad \cancel{x_1 + x_2}$

subject to

$$x_1 \geq 0$$
$$x_2 \geq 0$$

$$2x_2 - x_1 \leq 4$$
$$x_1 \leq 8$$
$$2x_2 + x_1 \leq 6$$

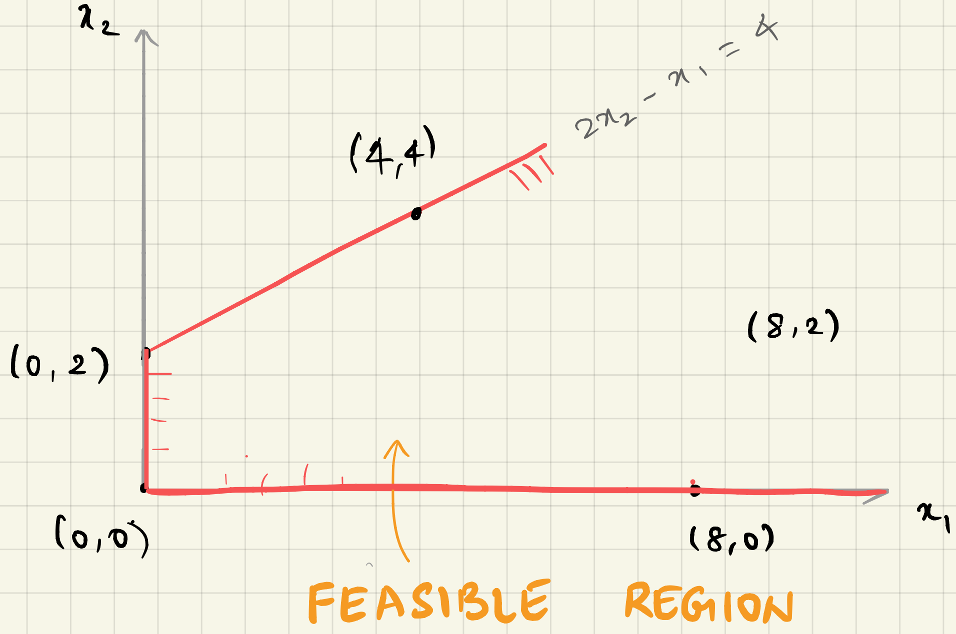




Maximize $\quad 2x_2 + x_1$ $\quad \cancel{x_1 + x_2}$

subject to

$$x_1 \geq 0$$
$$x_2 \geq 0$$
$$2x_2 - x_1 \leq 4$$
$$\cancel{x_1 \leq 8}$$
$$\cancel{2x_2 + x_1 \leq 6}$$



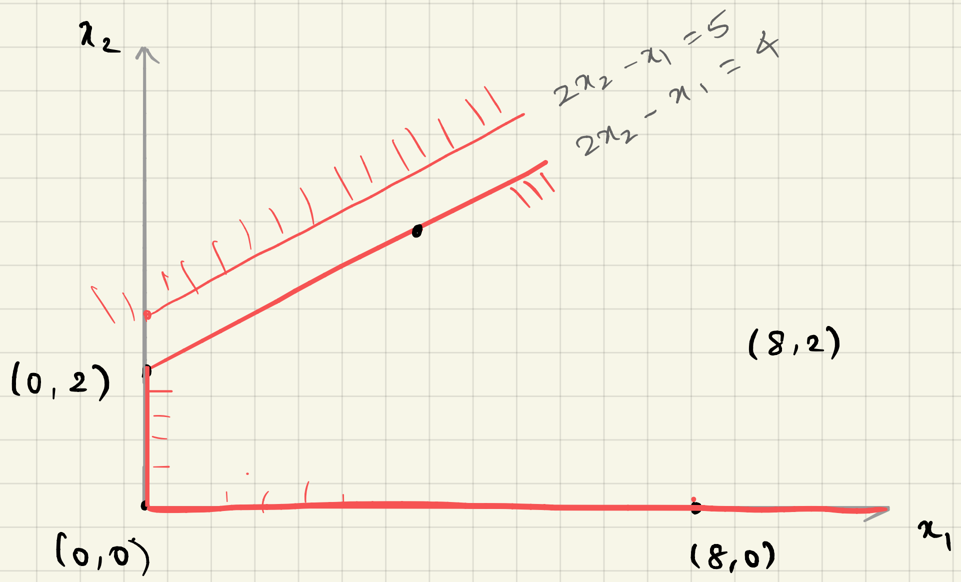
OPTIMUM does not exist

LP is unbounded.

Maximize $2x_2 + x_1$ ~~$x_1 + x_2$~~

Subject to

$$x_1 \geq 0$$
$$x_2 \geq 0$$
$$2x_2 - x_1 \geq 5$$
$$2x_2 - x_1 \leq 4$$
$$\cancel{x_1 \leq 8}$$
$$\cancel{2x_2 + x_1 \leq 6}$$



OPTIMUM does not exist

LP is infeasible

Four possible situations:

- 1. Unique optimum
- 2. Infinitely many optima
- 3. Unbounded LP
- 4. Infeasible LP

$\left.\begin{array}{l}\text{3. Unbounded LP}\\ \text{4. Infeasible LP}\end{array}\right\}$ optimum does not exist

These are the only possible situations (proof later in the course)

Later in the course:

- Extending these intuitions to higher dimensions
- Algorithms for finding the optimum

GOALS

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Part 2:

INTEGER LINEAR PROGRAMS (ILPs)

Maximize $c^T x$

subject to: $Ax \leq b$

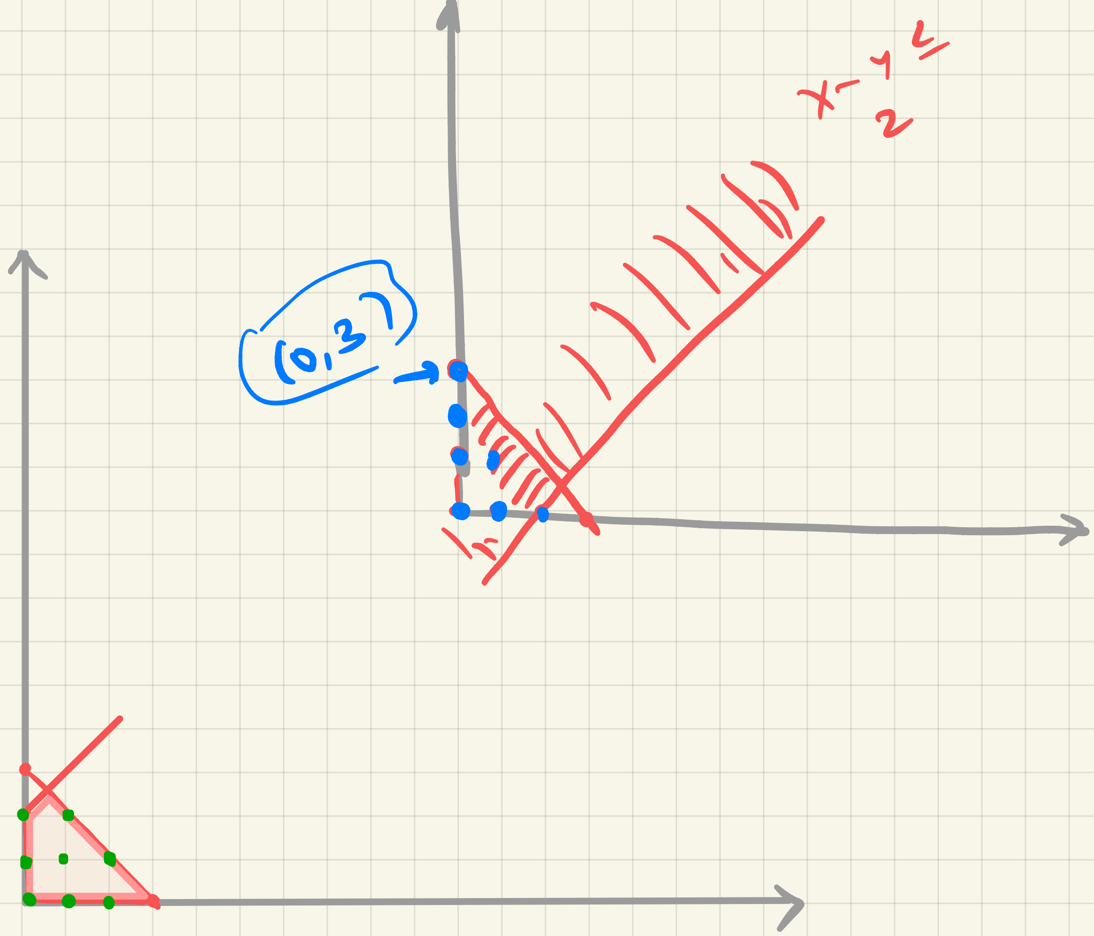
$x \in \mathbb{Z}^n$ additional constraint

Optimize the cost over all integral points in the feasible region.

EXAMPLE 1:

$$
\begin{aligned}
\text{Maximize} \quad & y \\
\text{subject to} \quad & x - y \leq 2 \\
& x + y \leq 3 \\
& x, y \geq 0 \\
& (x, y) \in \mathbb{Z}^2 \quad \leftarrow \text{integrality constraint}
\end{aligned}
$$

$$
\begin{aligned}
\text{Maximize} \quad & y \\
\text{subject to} \quad & y - x \leq 2 \\
& x + y \leq 3 \\
& x, y \geq 0 \\
& (x, y) \in \mathbb{Z}^2
\end{aligned}
$$



Optimum value $= 2$

attained at $(0, 2)$, $(1, 2)$

If $(x, y) \in \mathbb{R}^2$, optimum value $= 2.5$ attained at $(0.5, 2.5)$

Some facts:

- LPs can be efficiently solved.
  - PTIME algorithms
  - several heuristics studied

- ILPs are significantly harder to solve
  - NP-complete

Sometimes ILPs can be approximated using LPs.

Part 3:

LP relaxations of ILPs.

ILP:

Maximize $c^T x$

Subject to $Ax \leq b$

$x \in \mathbb{Z}^n$

Integrality constraint

$\rightarrow$

LP relaxation

Maximize $c^T x$

Subject to $Ax \leq b$

$x \in \mathbb{R}^n$

Integrality constraint is removed

ILP optimum value ? LP optimum value

$\leq$

Maximize y

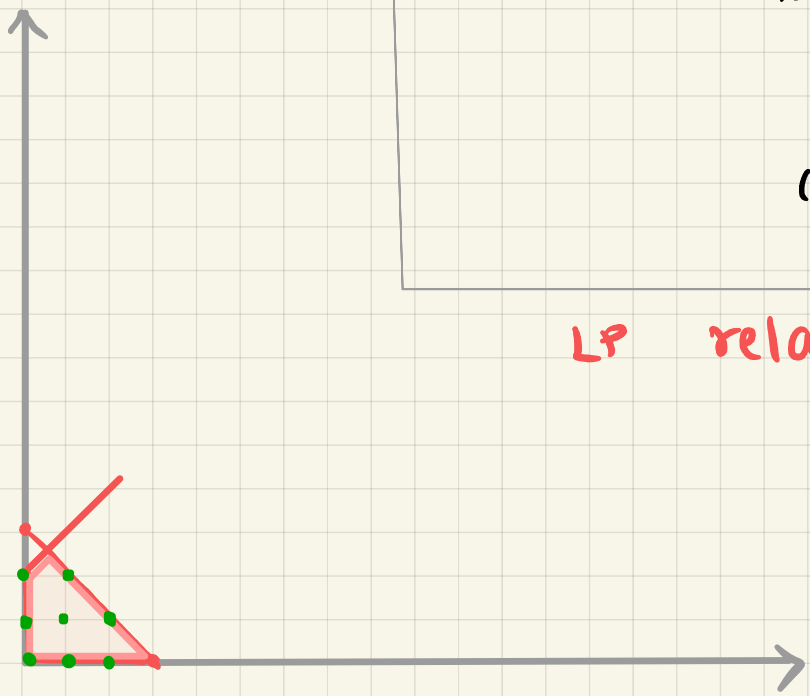
subject to
$$y - x \leq 2$$
$$x + y \leq 3$$
$$x, y \geq 0$$
$$(x, y) \in \mathbb{Z}^2$$

ILP

Maximize y

subject to
$$y - x \leq 2$$
$$x + y \leq 3$$
$$x, y \geq 0$$
$$(x, y) \in \mathbb{R}^2$$

LP relaxation



optimum value = 2

attained at $(0, 2)$, $(1, 2)$

If $\underbrace{(x, y) \in \mathbb{R}^2}_{\text{LP relaxation}}$, optimum value = 2.5 attained at $(0.5, 2.5)$

QUESTION:

(maximization problem)

Suppose the optimum of the LP relaxation occurs at an integral point.

What is the relationship between ILP optimum & LP (relaxation's) optimum ?

QUESTION: Suppose the optimum of the LP relaxation occurs at an integral point.

What is the relationship between ILP optimum & LP (relaxation's) optimum?

Answer: ILP optimum = LP optimum.

- ILP optimum $\leq$ LP optimum

LP optimum occurs at an integral point x^*

$\Rightarrow$ ILP optimum $\geq$ $c(x^*)$ [cost at x^*]

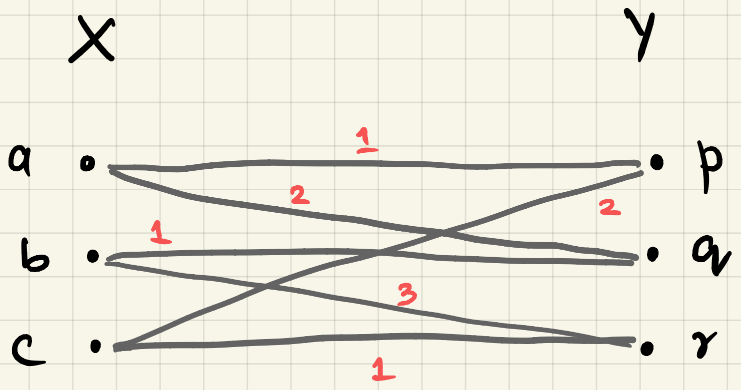
$\Rightarrow$ ILP optimum = LP optimum

Part 4:

Maximum Weight Matching in Bipartite graphs

Problem: Given a bipartite graph (X, Y, E) with $|X| = |Y|$

and $w: E \to \mathbb{R}_{\geq 0}$ (non-negative costs)

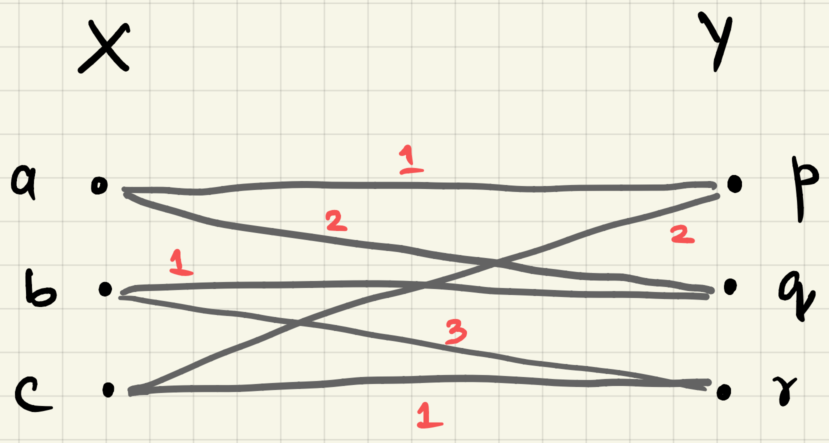


Matching: a subset $M \subseteq E$ such that

for every $x \in X$, exactly one edge in M is incident on x
and for every $y \in Y$ exactly one edge in M is incident on y

Maximum weight matching: Among all such matchings, find the one with the maximum weight.

Weight of a matching M is $\sum_{e \in M} w_e$



There are 2 matchings in this graph.

$a \overset{1}{\text{——}} p$	$a \overset{2}{\text{——}} q$
$b \overset{1}{\text{——}} q$	$b \overset{3}{\text{——}} r$
$c \overset{1}{\text{——}} r$	$c \overset{2}{\text{——}} p$
Weight: 3	Weight: 7 ← Max. weight matching

EXERCISE: Write an ILP for the maximum weight matching problem.

Given: a bipartite graph (X, Y, E) with $|X| = |Y|$ and weight function $w : E \mapsto \mathbb{R}_{\geq 0}$

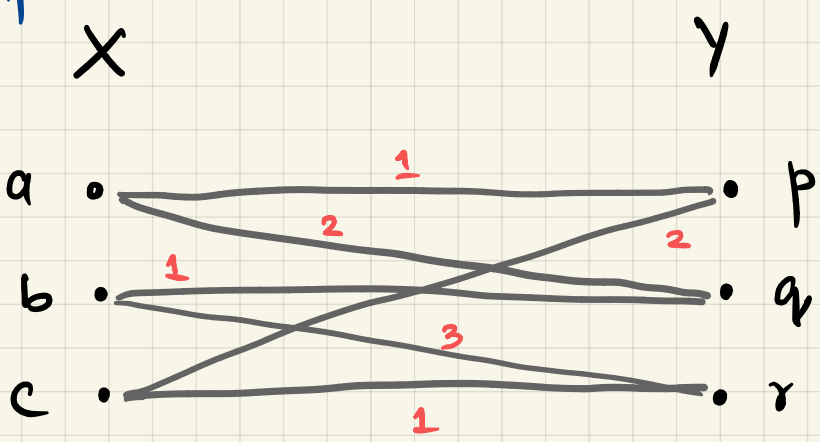
ILP for Maximum Weight matching

Step 1: Variables

x_e for each $e \in E$

Since we need to choose a subset of edges.

Example:



x_{ap}

x_{aq}

x_{bq}

x_{br}

x_{cp}

x_{cr}

ILP for Maximum Weight matching

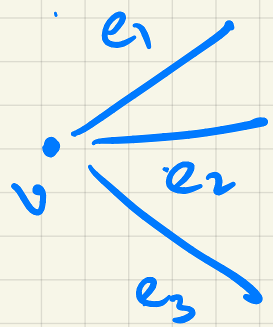
Step 1: Variables

x_e for each $e \in E$

Since we need to choose a subset of edges.

Step 2: Constraints

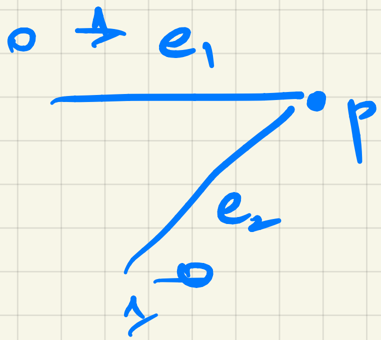
At each vertex only one edge of the chosen subset is incident.



$$\boxed{\forall v \in X \cup Y : \quad \sum_{e:\ e \text{ is incident on } v} x_e = 1}$$

$$\forall e : \quad 0 \leq x_e \leq 1$$

$$x_e \in \mathbb{Z}$$



ILP for Maximum Weight matching

Step 1: Variables

x_e for each $e \in E$

Since we need to choose a subset of edges.

Step 2: Constraints

At each vertex only one edge of the chosen subset is incident.

$$\boxed{\forall v \in X \cup Y : \sum_{\substack{e:\ e \text{ is} \\ \text{incident on } v}} x_e = 1}$$

$$\forall e: \quad 0 \leq x_e \leq 1$$

$$x_e \in \mathbb{Z}$$

$x_{ap} = 0$

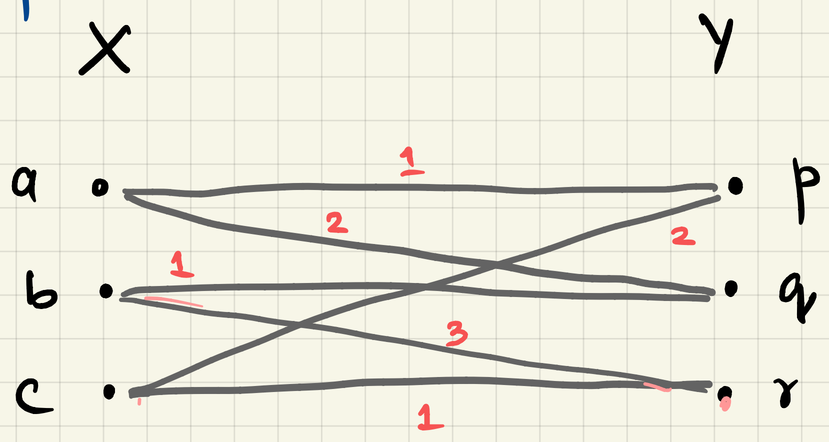
$x_{aq} = 1$ ✓

$\Rightarrow x_{bq} = 0$

$x_{br} = 1$ ✓

$x_{cp} = 1$

Example:



$$x_{ap} + x_{aq} = 1$$
$$x_{bq} + x_{br} = 1$$
$$x_{cp} + x_{cr} = 1$$
$$x_{ap} + x_{cp} = 1$$
$$x_{aq} + x_{bq} = 1$$
$$x_{br} + x_{cr} = 1$$

$$0 \leq x_{ap}, \ldots, x_{cr} \leq 1$$

$$x_{ap}, \ldots, x_{cr} \in \mathbb{Z}$$

ILP for Maximum Weight matching

Step 1: Variables

x_e for each $e \in E$

Since we need to choose a subset of edges.

Step 2: Constraints

At each vertex only one edge of the chosen subset is incident.

$$\forall v \in X \cup Y: \quad \sum_{\substack{e:\, e \text{ is} \\ \text{incident on } v}} x_e = 1$$

$$\forall e: \quad 0 \leq x_e \leq 1$$

$$x_e \in \mathbb{Z}$$

Step 3: Objective

Maximize cost of the chosen subset

$$\text{Maximize} \sum_{e \in E} w_e x_e$$

ILP for Maximum Weight matching

Step 1: Variables

x_e for each $e \in E$

Since we need to choose a subset of edges.

Step 2: Constraints

At each vertex only one edge of the chosen subset is incident.

$$\forall v \in X \cup Y: \quad \sum_{e:\, e \text{ is incident on } v} x_e = 1$$

$$\forall e: \quad 0 \le x_e \le 1$$

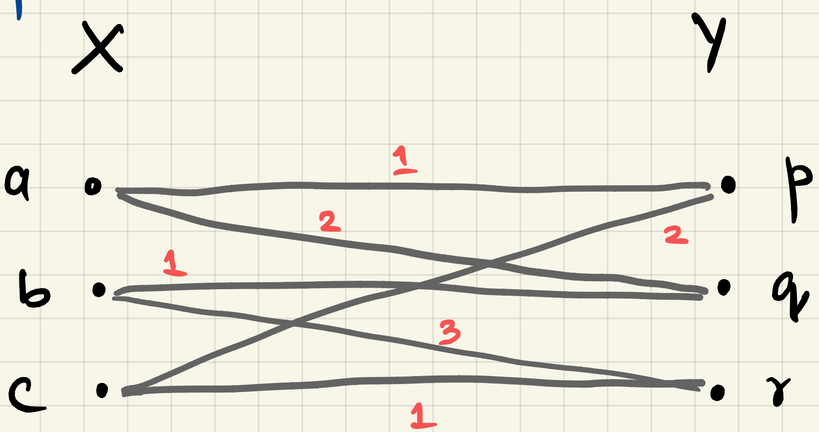
$$x_e \in \mathbb{Z}$$

Step 3: Objective

Maximize cost of the chosen subset

$$\text{Maximize} \sum_{e \in E} w_e \, x_e$$

Example:



Maximize: $1 \cdot x_{ap} + 2 \cdot x_{aq} + 1 \cdot x_{bq} + 3 \cdot x_{br} + 2 \cdot x_{cp} + 1 \cdot x_{cr}$

ILP for maximum weight matching:

Maximize $\displaystyle\sum_{e \in E} w_e \, x_e$

Subject to

$$\sum_{e:\ e \text{ incident on } v} x_e = 1 \qquad \forall v \in X \cup Y$$

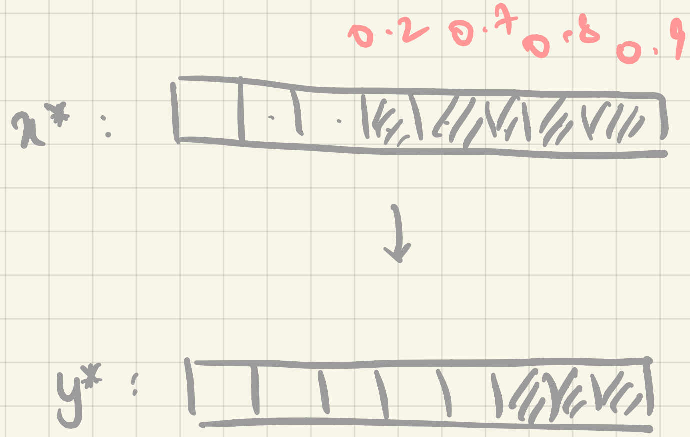
$$\left.\begin{aligned} 0 \le x_e \le 1 \\ x_e \in \mathbb{Z} \end{aligned}\right\} \forall e.$$

Claim: ILP optimum = LP optimum for above LP!

$$\text{Maximize} \quad \sum_{e \in E} w_e \, x_e$$

$$\text{Subject to} \quad \sum_{\substack{e:\ e \\ \text{incident on} \\ v}} x_e = 1 \qquad \forall v \in X \cup Y$$

$$\left.\begin{array}{l} 0 \leq x_e \leq 1 \\ x_e \in \mathbb{Z} \end{array}\right\} \forall e.$$



Claim: ILP optimum = LP optimum for above LP!

Idea of proof: Suppose x^* is optimum of LP, and has non-integral values.

We can find y^* s.t. $\text{cost}(y^*) = \text{cost}(x^*)$

and y^* has strictly fewer non-integral coordinates.

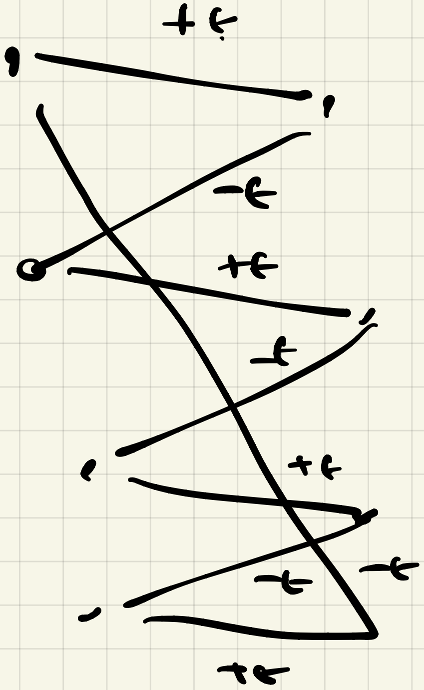
x^* — Cycle is $e_1\ e_2\ e_3 \ldots e_{2k}$

y^* :

$$y^*(x_{e_i}) = x^*(x_{e_i}) + \epsilon \quad i \text{ is odd}$$

$$= x^*(x_{e_i}) - \epsilon \quad \text{if } i \text{ is even}$$

$$= x^*(x_e) \quad \text{for edges not in cycle.}$$



- y^* is a feasible solution.

- as every constraint is still satisfied.

$$\text{Cost}(y^*) = \sum w_e y_e^*$$

out of cycle. cycle - odd cycle - even

$$\sum w_e x_e^* \qquad \sum w_e (x_e^* + \epsilon) \qquad \sum w_e (x_e^* - \epsilon)$$

$$\text{Cost} = \text{Cost}(x^*) + \epsilon \Big[\underbrace{\sum w_e}_{\text{odd no.s edges in cycle}} - \underbrace{\sum w_e}_{\text{even no. edges.}} \Big]$$

$$= \text{Cost}(x^*) + \epsilon \, \Delta$$

$$\text{Cost} = \text{Cost}(x^*) + \epsilon\Big[\underbrace{\sum w_e}_{\substack{\text{odd no. of}\\ \text{edges in}\\ \text{cycle}}} - \underbrace{\sum w_e}_{\substack{\text{even no.}\\ \text{edges.}}}\Big]$$

$$\text{Cost}(y^*) = \text{Cost}(x^*) + \epsilon\,\boxed{\Delta}$$

- Suppose $\Delta > 0$: $\text{Cost}(y^*) > \text{Cost}(x^*)$

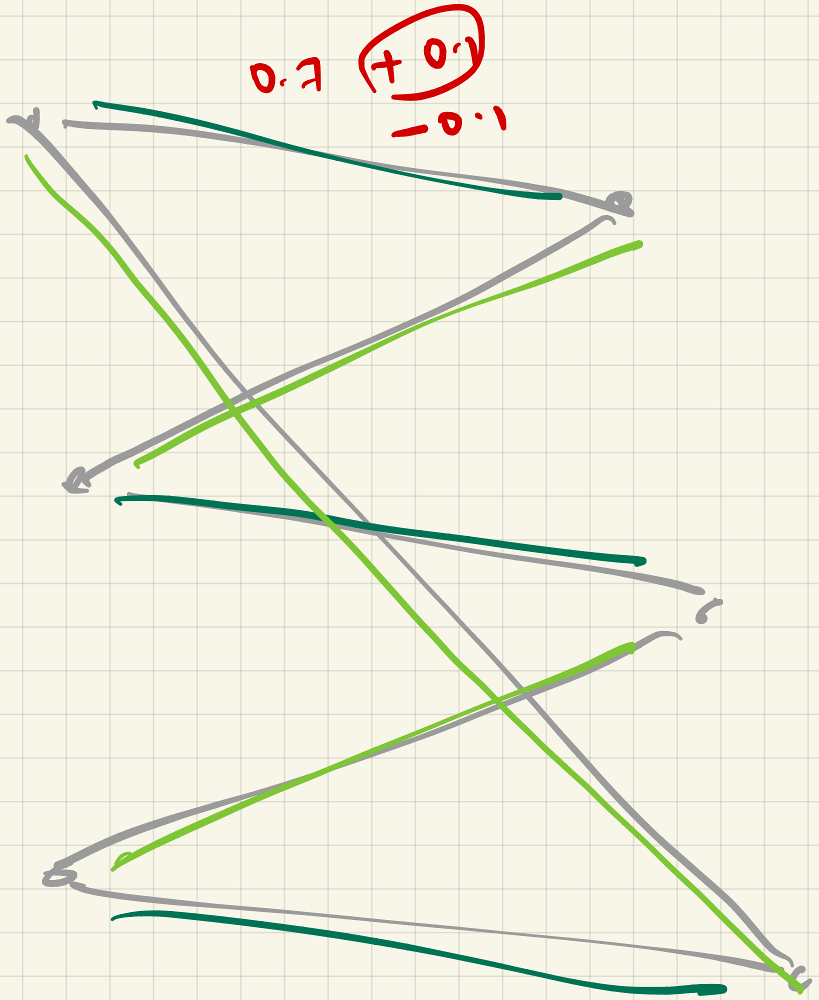
  - contradicts the hyp. that x^* is optimum.

- Suppose $\Delta < 0$: We can find ϵ and add to all even no. edges and subtract from all odd edges.

  $\Rightarrow \text{Cost}(y^*) > \text{Cost}(x^*)$. contradiction.

$\Rightarrow \Delta = 0 \Rightarrow \text{Cost}(y^*) = \text{Cost}(x^*)$

$\Delta < 0$



$\Delta = 0$

$$x^* \longrightarrow y^*$$

$$\text{Cost}(y^*) = \text{Cost}(x^*)$$

- If we choose the ϵ with the maximum modulus, atleast one of the coordinates becomes integral

$\Rightarrow$ y^* will have strictly fewer non-integral coordinates.

$$x^* \longrightarrow y^* \longrightarrow y_1^* \longrightarrow y_2^* \quad \cdots \longrightarrow z^*$$

only integers.

- In each of the transformations, the cost does not change.

$\Rightarrow$ From the LP optimum, we can get an integral valuation with the same cost.

$\Rightarrow$ ILP optimum $=$ LP optimum.

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