LINEAR OPTIMIZATION

LECTURE 17

COMPLEMENTARY SLACKNESS

maximize $c^T x$	minimize $b^T y$
subj. to $Ax \leq b$	subj. to $A^T y = c$
	$y \geq 0$

Theorem: Let x_0, y_0 be feasible solutions of primal and dual respectively.

Then: x_0 and y_0 are optima $\Leftrightarrow$ $c^T x_0 = b^T y_0$

Proof: $(\Rightarrow)$ Suppose x_0 and y_0 are optima of primal and dual resp.

From duality theorem, optimum-cost (primal) = optimum-cost (dual)

since both optima exist.

$$\Rightarrow \quad c^T x_0 = b^T y_0$$

$(\Leftarrow)$ We are given that x_0 is a feasible soln. of primal

y_0 feasible soln. of dual

$+ \quad c^T x_0 = b^T y_0$

i) Both primal and dual have optima, since both of them are feasible.

ii) $c^T x \leq b^T y_0$ $\forall$ feasible soln. x of primal

(weak duality)

Since $c^T x_0 = b^T y_0$, x_0 is an optimum of primal

$(\Leftarrow)$ We are given that x_0 is a feasible soln. of primal
y_0 feasible soln. of dual
+ $c^T x_0 = b^T y_0$

i) Both primal and dual have optima, since both of them are feasible.

ii) $c^T x \leq b^T y_0$ $\forall$ feasible soln. x of primal (weak duality)

Since $c^T x_0 = b^T y_0$, x_0 is an optimum of primal

iii) Similarly: $c^T x_0 \leq b^T y$ $\forall y$ which are feasible in dual.

Since $c^T x_0 = b^T y_0$, y_0 is an optimum of dual.

maximize $c^T x$

subj. to $Ax \leq b$

minimize $b^T y$

subj. to $A^T y = c$

$y \geq 0$

Theorem (COMPLEMENTARY SLACKNESS CONDITION):

x_0: primal feasible solution $\quad$ y_0: dual feasible solution

Then: $\boxed{c^T x_0 = b^T y_0 \text{ iff } (y_0)_i > 0 \Rightarrow A_i x_0 = b_i \quad \forall i \in \{1, 2, \ldots, m\}}$

Dual variable slack $\Rightarrow$ Primal inequality tight

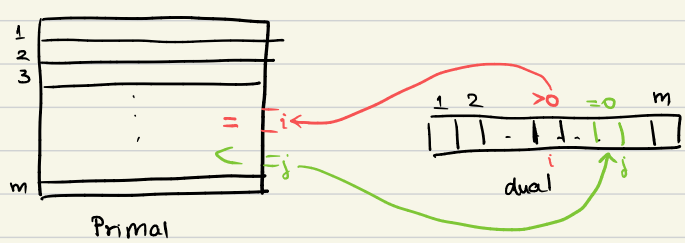
Primal inequality slack $\Rightarrow$ dual variable tight

$A_i x < b_i$

↳ i is 'slack'

$A_i x = b_i$

↳ i is 'tight'



- Gives another criteria to check for optimum.

Proof. $(\Rightarrow)$ Suppose $c^T x_0 = b^T y_0$

To show:

$$\boxed{(y_0)_i > 0 \Rightarrow A_i x_0 = b_i \quad \forall i \in \{1, 2, \dots, m\}}$$

$$c^T x_0 = c_1 (x_0)_1 + c_2 (x_0)_2 + \dots + c_n (x_0)_n$$

We know $A^T y_0 = c$

$$A^1 y_0 = c_1 \qquad A^i : i^{th} \text{ column of } A$$

$$\vdots$$

$$A^i y_0 = c_i$$

$$\vdots$$

$$A^n y_0 = c_n$$

$$\therefore c^T x_0 = (A^1 y_0) \cdot (x_0)_1 + (A^2 y_0)(x_0)_2 + \dots + (A^n y_0)(x_0)_n$$

$$= (a_{11} (y_0)_1 + a_{21} (y_0)_2 + \dots + (a_{m1})(y_0)_m)(x_0)_1$$

$$+$$

$$\vdots$$

$$(a_{1n} (y_0)_1 + a_{2n} (y_0)_2 + \dots + a_{mn} (y_0)_m)(x_0)_n$$

Rearranging:

$$= (y_0)_1 [a_{11} (x_0)_1 + \dots + a_{1n} (x_0)_n]$$

$$+ \dots + (y_0)_m [a_{m1} (x_0)_1 + \dots + a_{mn} (x_0)_n]$$

$$\therefore c^T x_0 = (A^1 y_0)(x_0)_1 + (A^2 y_0)(x_0)_2 + \cdots + (A^n y_0)(x_0)_n$$

$$= (a_{11}(y_0)_1 + a_{21}(y_0)_2 + \cdots + (a_{m1})(y_0)_m)(x_0)_1$$

$$+$$

$$\vdots$$

$$(a_{1n}(y_0)_1 + a_{2n}(y_0)_2 + \cdots + a_{mn}(y_0)_m)(x_0)_n$$

Rearranging:

$$= (y_0)_1 [a_{11}(x_0)_1 + \cdots + a_{1n}(x_0)_n]$$

$$+ \cdots + (y_0)_m [a_{m1}(x_0)_1 + \cdots + a_{mn}(x_0)_n]$$

$$c^T x_0 = (y_0)_1 [A_1 x_0] + (y_0)_2 [A_2 x_0] + \cdots + (y_0)_m [A_m x_0]$$

We are given $c^T x_0 = b^T y_0$

$$= (y_0)_1 b_1 + \cdots + (y_0)_m b_m$$

$$\therefore (y_0)_1 [A_1 x_0] + \cdots + (y_0)_m [A_m x_0] = (y_0)_1 b_1 + \cdots + (y_0)_m b_m$$

$$\Rightarrow \sum_{i=1}^{n} (y_0)_i [b_i - A_i x_0] = 0$$

$$\sum_{i=1}^{m} (y_0)_i \left[b_i - A_i x_0 \right] = 0$$

$(y_0)_i \geq 0$, similarly $b_i - A_i x_0 \geq 0$

Since we have a sum of non-negative terms equal to 0, each of the terms has to be 0.

$$\therefore \quad (y_0)_i \left[b_i - A_i x_0 \right] = 0 \quad \forall i \in \{1, \dots, m\}$$

$$\Rightarrow \quad (y_0)_i > 0 \quad \Rightarrow \quad A_i x_0 = b_i$$

Theorem (COMPLEMENTARY SLACKNESS CONDITION):

x_0: primal feasible solution $\qquad$ y_0: dual feasible solution

Then: $\boxed{c^T x_0 = b^T y_0 \text{ iff } (y_0)_i > 0 \Rightarrow A_i x_0 = b_i \quad \forall i \in \{1, 2, \ldots, m\}}$

Proof of: $\Leftarrow$:

Suppose $(y_0)_i > 0 \Rightarrow A_i x_0 = b_i$

To show: $c^T x_0 = b^T y_0$

$$c^T x_0 = c_1 (x_0)_1 + c_2 (x_0)_2 + \ldots + c_n (x_0)_n$$

$$= (A^1 y_0)(x_0)_1 + \ldots + (A^n y_0)(x_0)_n$$

$$c^T x_0 - b^T y_0 = \sum_{i=1}^{n} (A^i y_0)(x_0)_i - \sum_{i=1}^{m} b_i (y_0)_i$$

Rearranging as before to collect $(y_0)_i$ terms together:

$$c^T x_0 = [a_{11} (y_0)_1 + a_{21} (y_0)_2 + \ldots + a_{m1} (y_0)_m] (x_0)_1$$

$$+ \ldots$$

$$[a_{1n} (y_0)_1 + \ldots \quad a_{mn} (y_0)_m] (x_0)_n$$

$$c^T x_0 - b^T y_0 = \sum_{i=1}^{n} (A^i y_0)(x_0)_i - \sum_{i=1}^{m} b_i (y_0)_i$$

Rearranging as before to collect $(y_0)_i$ terms together

$$c^T x_0 = [a_{11}(y_0)_1 + a_{21}(y_0)_2 + \cdots + a_{m1}(y_0)_m](x_0)_1$$

$$+ \quad \cdots$$

$$[a_{1n}(y_0)_1 + \cdots \qquad a_{mn}(y_0)_m](x_0)_n$$

$$= (y_0)_1 [A_1 x_0] + (y_0)_2 [A_2 x_0] + \cdots + (y_0)_m (A_m x_0)$$

$$\therefore \quad c^T x_0 - b^T y_0 = \sum_{i=1}^{m} (A_i x_0)(y_0)_i - \sum_{i=1}^{m} b_i (y_0)_i$$

$$= \sum_{i=1}^{m} (y_0)_i [A_i x_0 - b_i]$$

From hypothesis: $(y_0)_i > 0 \implies A_i x_0 = b_i$

Hence $c^T x_0 - b^T y_0 = 0$

$$c^T x_0 = b^T y_0$$

COMPLEMENTARY SLACKNESS FOR OTHER PRIMAL - DUAL PAIRS

Primal	Dual
maximize $c^T x$	minimize $b^T y$
subj. to $Ax \leq b$, $x \geq 0$	subj. to $A^T y \geq c$, $y \geq 0$

Theorem: Let x_0, y_0 be feasible solutions of primal and dual respectively.

Then: $c^T x_0 = b^T y_0$ iff

i) $(y_0)_i > 0 \Rightarrow A_i x_0 = b_i \quad \forall i \in \{1, 2, \ldots, m\}$

AND

ii) $(x_0)_j > 0 \Rightarrow A_j^T y_0 = c_j \quad \forall j \in \{1, 2, \ldots, n\}$

Proof: $(\Leftarrow)$ To show $c^T x_0 - b^T y_0 = 0$

$$c^T x_0 = c_1 (x_0)_1 + c_2 (x_0)_2 + \cdots + c_n (x_0)_n$$

Notice that $(x_0)_j = 0$ or $(x_0)_j > 0$

When $(x_0)_j > 0$, we have $c_j = A_j^T y_0$ (from (ii))

$$\therefore \; c^T x_0 = (A_1^T y_0)(x_0)_1 + \cdots + (A_n^T y_0)(x_0)_n$$

$$c^T x_0 = (A_1^T y_0)(x_0)_1 + \cdots + (A'_n y_0)(x_0)_n$$

Rearranging in terms of $(y_0)_1, (y_0)_2, \ldots (y_0)_m$.

$Ax \leq b$

$(y_0)_1 \times$

$(y_0)_2$

$\vdots$

$(y_0)_m \times$

$$= (y_0)_1 A_1 x_0 + (y_0)_2 A_2 x_0 + \cdots + (y_0)_m A_m x_0$$

From (i), we can rewrite this as:

$$= (y_0)_1 b_1 + (y_0)_2 b_2 + \cdots + (y_0)_m b_m$$

$$= b^T y_0$$

$$\therefore c^T x_0 = b^T y_0.$$

$\Rightarrow$ Suppose $c^T x_0 = b^T y_0$

We have $c \leq A^T y$

$$c \leq A^T y_0$$

$\therefore \quad c_1 \leq A^1 y_0 \quad \ldots \quad c_n \leq A^n y_0$

$c^T x_0 \leq (A^1 y_0)(x_0)_1 + \cdots + (A^n y_0)(x_0)_n$

Rearranging $\leq (y_0)_1 (A_1 x_0) + \ldots + (y_0)_m (A_m x_0)$

$\leq (y_0)_1 b_1 + \ldots + (y_0)_m b_m$ (as x_0 is feasible in primal)

But from hypothesis: $c^T x_0 = b^T y_0$

$\therefore \quad c^T x_0 = \sum_{i=1}^{m} (y_0)_i (A_i x_0) = b^T y_0$

$\therefore \quad b^T y_0 - \sum_{i=1}^{m} (A_i x_0)(y_0)_i = 0$

$$\sum_{i=1}^{m} (b_i - A_i x_0)(y_0)_i = 0$$

Since each term $(b_i - A_i x_0)(y_0)_i \geq 0$, we have :

i) $(y_0)_i > 0 \implies A_i x_0 = b_i$

$$b^T y_0 = b_1 (y_0)_1 + \cdots + b_m (y_0)_m = \sum_{i=1}^{m} b_i (y_0)_i \qquad Ax \leq b$$

$$\geq \sum_{i=1}^{m} (A_i x_0)(y_0)_i \qquad \text{since } x_0 \text{ is feasible in primal}$$

Rearrange to bring $(x_0)_i$ terms together:

$$= \sum_{i=1}^{n} (x_0)_i \, A_i^T y_0$$

$$\therefore \quad b^T y_0 \geq \sum_{i=1}^{n} (A_i^T y_0)(x_0)_i \geq \sum_{i=1}^{n} c_i (x_0)_i \qquad \text{as } y_0 \text{ is feasible in dual.}$$

$$= c^T x_0$$

But since $b^T y_0 = c^T x_0$, we have

$$b^T y_0 = \sum_{i=1}^{n} (A_i^T y_0)(x_0)_i = c^T x_0$$

$$\therefore \quad \sum_{i=1}^{n} (A_i^T y_0)(x_0)_i - c^T x_0 = 0$$

$$\sum_{i=1}^{n} [A_i^T y_0 - c_i](x_0)_i = 0$$

Once again, each term on the LHS is non-negative.

$\therefore$ As the sum is 0, we have:

$$[A_i^T y_0 - c_i](x_0)_i = 0 \quad \forall i \in \{1, \ldots, n\}$$

i.e. $(x_0)_i > 0 \implies A_i^T y_0 = c_i$

Primal: maximize $c^T x$

subj. to: $A_1 x \sim_1 b_1$ $\quad\quad \sim_i \in \{\leq, \geq, =\}$

$A_2 x \sim_2 b_2$

$\vdots$

$A_m x \sim_m b_m$

$x_1 \sim'_1 0$

$x_2 \sim'_2 0$ $\quad\quad \sim'_i \in \{\leq, \geq, \gtrless\}$

$\vdots$ $\quad\quad\quad\quad\quad\quad \gtrless$: unrestricted.

$x_n \sim'_n 0$

- General form primal will have a corresponding dual.

Theorem: (Complementary slackness for general primal-dual pairs)

Let x_0, y_0 be feasible solutions of primal and dual respectively.

Then: $c^T x_0 = b^T y_0$ iff:

i) $(y_0)_i \, (A_i x_0 - b_i) = 0 \quad \forall i \in \{1, 2, \dots, m\}$

ii) $(x_0)_j \, (c_j - A_j^T y_0) = 0 \quad \forall i \in \{1, 2, \dots, n\}$

Proof: Exercise.