# LINEAR OPTIMIZATION

LECTURE 16

10/6/2021

# A GEOMETRIC INTERPRETATION OF DUALITY

**REFERENCE:** Lecture 11 from Prof. Sundar Vishwanathan's notes on Linear Optimization

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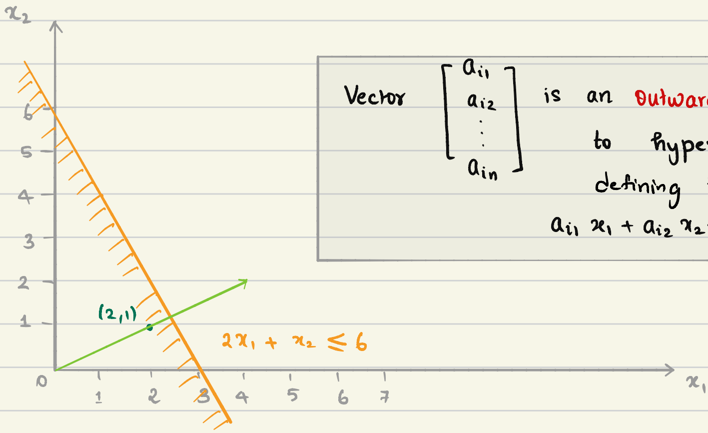
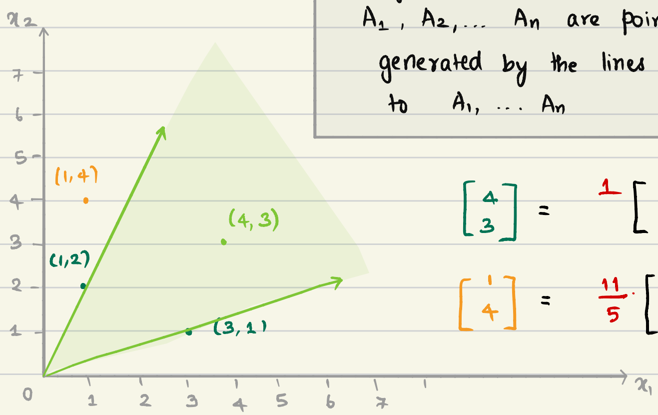
## Some basic observations:

1)




Vector $\begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}$ is an outward normal to hyperplane defining the half-space $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i$

2)




Non-negative combinations of vectors $A_1, A_2, \dots A_n$ are points in the cone generated by the lines joining origin to $A_1, \dots A_n$

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \frac{11}{5} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{2}{5} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Consider an LP:

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \\ & x \text{ unrestricted} \end{aligned}$$

$A: m \times n$
$c: n \times 1$
$x: n \times 1$
$b: m \times 1$

Assume that the LP is non-degenerate, and has an optimum at an extreme point

Extreme point: a feasible solution that:

- is obtained as the intersection of $n$ linearly independent hyperplanes from $A_1 x = b_1$, $A_2 x = b_2, \ldots, A_m x = b_m$

$$A' x = b' \quad (\text{the set of } n \text{ hyperplanes})$$
$$A'' x < b'' \quad (\text{others})$$

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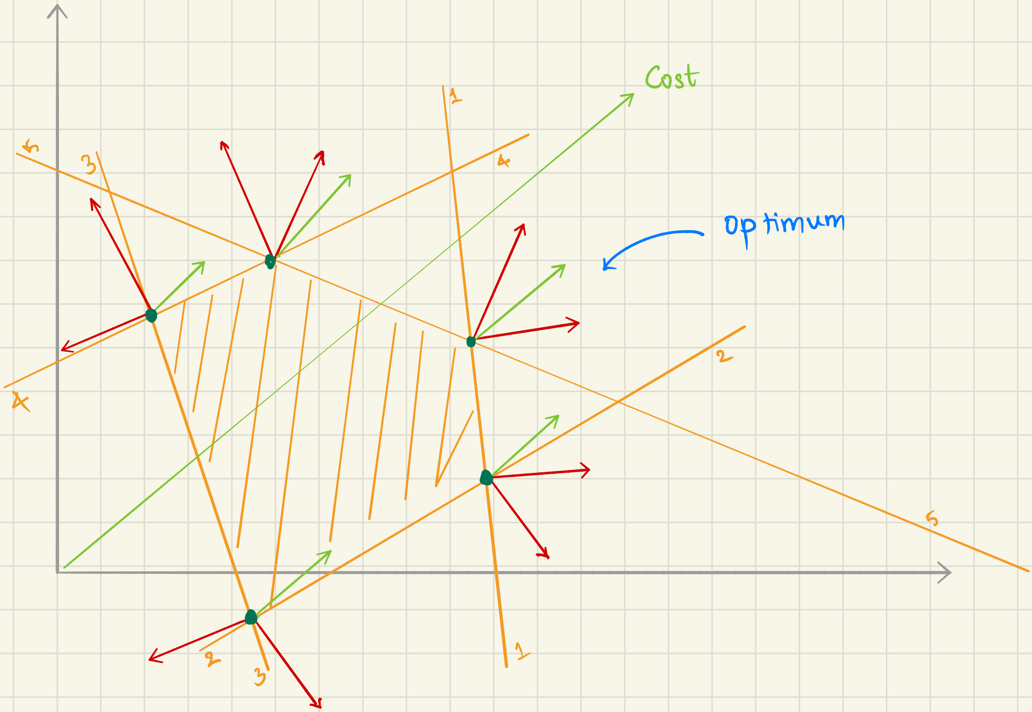
**Theorem:** An extreme point defined by $A'x = b'$, $A''x < b''$ is an optimum

iff

the cost vector '$c$' can be written as a non-negative combination of $A'_1, A'_2, \ldots A'_n$

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## KEY OBSERVATION

At optimum extreme point, cost decreases along all directions to the neighbouring extreme points.

# CHARACTERIZING NEIGHBOURING DIRECTIONS FROM AN EXTREME POINT:

$u_0$ : extreme point

$$A'x = b'$$
$$A''x < b''$$

$$\begin{matrix} A_1' & b_1' \\ A_2' & b_2' \\ \vdots & \vdots \\ A_n' & b_n' \end{matrix}$$

Suppose $u_i$ is a point such that: $A_j' u_i = b_j' \quad \forall j \neq i$

$$A_i' u_i < b_i'$$

For example: $u_1$ :

$$A_1' u_1 < b_1'$$
$$\left.\begin{matrix} A_2' u_2 = b_2' \\ \vdots \\ A_n' u_n = b_n' \end{matrix}\right\} \text{tight}$$

- From each extreme point, there are 'n' rays moving out to its neighbours.

Vectors $u_1 - u_0$ $\quad u_2 - u_0$ $\quad \ldots \quad u_n - u_0$ characterize these directions.

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- Note that:

$$A'(u_1 - u_0) = \begin{bmatrix} A_1' u_1 - A_1' u_0 \\ A_2' u_1 - A_2' u_0 \\ \vdots \\ A_n' u_1 - A_n' u_0 \end{bmatrix} = \begin{bmatrix} \text{some negative value} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- By sufficient scaling, we can assume that $A_1'(u_1 - u_0) = -1$

Similarly $A_i'(u_i - u_0) =$

$$A'(u_i - u_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ place}$$

We have now 'n' vectors:

$$U = \begin{bmatrix} u_1 - u_0 & u_2 - u_0 & \cdots & u_n - u_0 \end{bmatrix}$$

s.t.

$$A' U = \begin{bmatrix} -1 & 0 & & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & 0 & & \vdots \\ 0 & \vdots & & \\ & 0 & & -1 \end{bmatrix}$$

This implies:

$$\boxed{U = -(A')^{-1}}$$

**Theorem:** Directions to neighbours from an extreme point defined by $A'x = b'$, $A''x < b''$ are given by columns of $-(A')^{-1}$.

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At optimum, cost decreases along all its neighbours.

Suppose $u_0$ is optimum.

$$c^T (u_i - u_0) \leq 0$$

$$\therefore \quad c^T \left[-(A')^{-1}\right]_i \leq 0 \quad \forall i \in \{1 \ldots n\}$$

$i^{th}$ column of $-(A')^{-1}$

$$\therefore \quad \boxed{c^T (A')^{-1}_i \geq 0 \quad \forall i \in \{1 \ldots n\}} \quad (*)$$

Why $c^T (x_i - x_0) \leq 0$ ?

There is some 't' s.t.

$x_0 + t(x_i - x_0)$ is an extreme point of $Ax \leq b$.

$A_1' (x_i - x_0) = 0$

$A_2'$

$\vdots$

$A_i' (x_i - x_0) = <$

$\vdots$

$A_n'$

$A''$

Consider the points: $x_0 + t(x_i - x_0)$

- At $t=0$, we get $x_0$.
- On increasing $t$, what happens?

- We can find a $t > 0$ s.t. $x_0 + t(x_i - x_0)$ is feasible.

  ie, $A(x_0 + t(x_i - x_0)) \leq b$

- If $c^T(x_i - x_0) > 0$, then $c^T(x_0 + t(x_i - x_0)) > c^T x_0$

  — contradiction that $x_0$ is optimum.

## Writing the cost vector as a linear combination of rows of $A'$:

$A'x = b'$ contains 'n' linearly independent hyperplanes:

$$A_1' x = b_1'$$
$$\vdots$$
$$A_n' x = b_n'$$

- Vectors: $A_1', \ldots A_n'$ are linearly independent.

$\therefore$ Cost $c$ can be written as a linear combination of them

$$c = \alpha_1 . A_1' + \alpha_2 A_2' + \ldots + \alpha_n A_n'$$

$n \times 1$: column vectors.

$c^T (A')^{-1}$

Rewriting: $c^T = \alpha_1 A_1'^T + \alpha_2 A_2'^T + \ldots + \alpha_n A_n'^T$

- Note that: $c^T . (A')^{-1}_1 = \alpha_1$

$$\vdots$$

$$c^T . (A')^{-1}_n = \alpha_n$$

$$A'(A')^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

From $(*)$, $\alpha_1, \alpha_2, \ldots \alpha_n \geq 0$

- Proves that at optimum, cost can be written as a non-negative linear combination of the outward normals of the 'n' hyperplanes defining the extreme point!

Rewriting: $c^T = \alpha_1 A_1'^T + \alpha_2 A_2'^T + \cdots + \alpha_n A_n'^T$

Multiplying with $(A')^{-1}$ on both sides.

$$c^T (A')^{-1} = \alpha_1 A_1'^T (A')^{-1} + \cdots + \alpha_n (A_n')^T (A')^{-1}$$

$$c^T (A')^{-1} : (1 \times n) \times (n \times n)$$

$$= [\beta_1 \ \beta_2 \ \cdots \ \beta_n]$$

where $\beta_i \geq 0 \quad \forall i \in \{1, \ldots n\}$

$$\alpha_1 (A_1')^T (A')^T = \underset{A_1'}{[\cdot \ \cdot \ \cdot \ \cdot \ \cdot]} \begin{bmatrix} & (A')^{-1} \\ & \end{bmatrix}$$

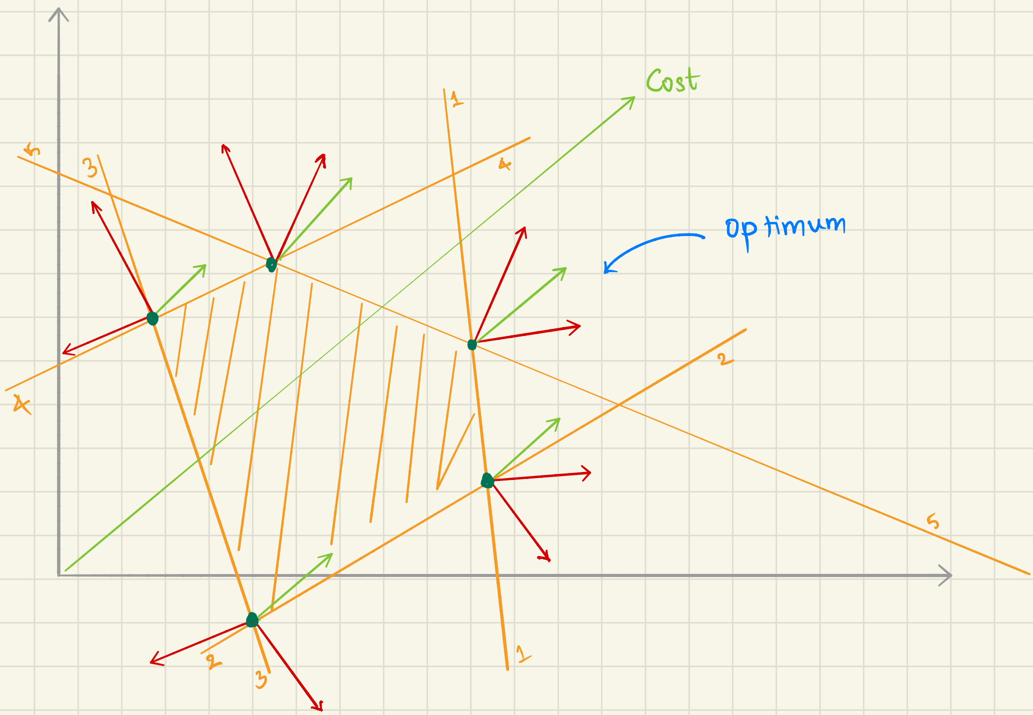
$$= \alpha_1 [1 \ 0 \ 0 \ \ldots \ 0]$$

$\vdots$

$$\alpha_i (A_i')^T (A')^T = \alpha_i [0 \ 0 \ \ldots \ \underset{\downarrow \\ i^{th} \text{ place.}}{1} \ \ldots \ 0 \ 0]$$

$$\therefore [\beta_1 \ \beta_2 \ \cdots \ \beta_n] = [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n]$$

# GEOMETRIC INTERPRETATION OF DUALITY




Given LP maximize $c^T x$ subject to $Ax \leq b$

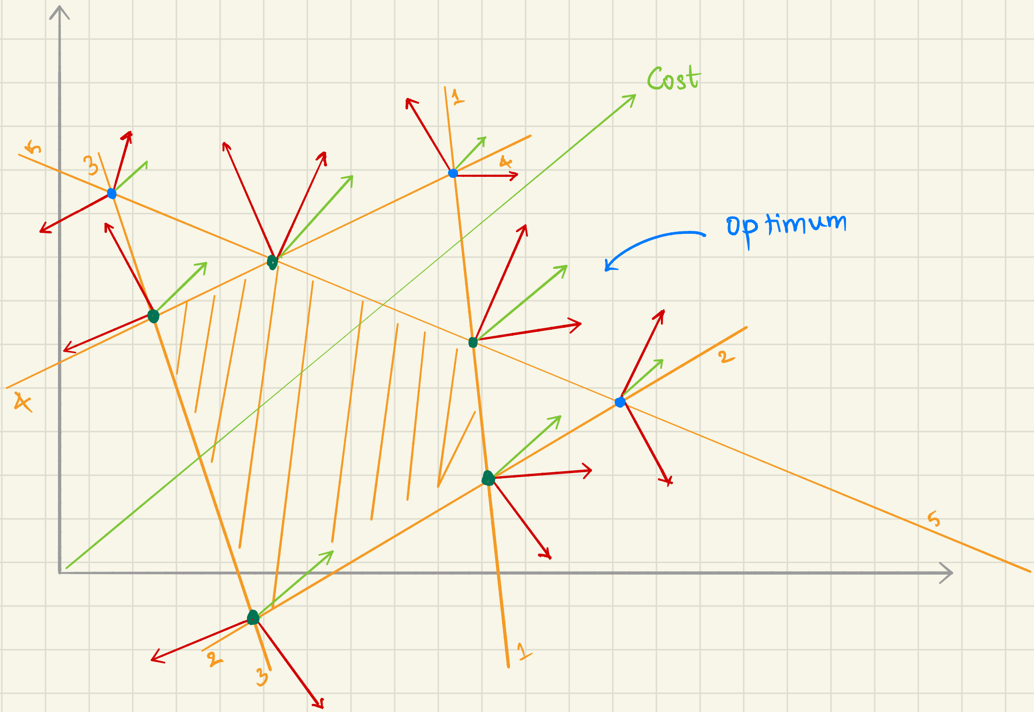
- non-degenerate, has optimum

Intersection points: Points satisfying
$$A_{i_1} x = b_{i_1}$$
$$A_{i_2} x = b_{i_2}$$
$$\vdots$$
$$A_{i_n} x = b_{i_n}$$

for some $n$ linearly independent rows from $Ax \leq b$

- These points need not satisfy the other inequalities.

Extreme points: Intersection points that are also feasible.




Define $F :=$ set of intersection points such that cost vector 'c' is a non-negative combination of the outward normals of the defining hyperplanes

Point $x$ satisfying $A_{i_1} x = b_1, \dots, A_{i_n} x = b_n$ $\in F$ if $c^T = \alpha_1 A_{i_1} + \alpha_2 A_{i_2} + \dots + \alpha_n A_{i_n}$

such that $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_n} \geq 0$

**Theorem:** Optimum of LP is a point in F with minimum cost.

**To Prove:** For every point $x \in F$, $c^T x \geq c^T x_0$, where $x_0$ is the optimum of the LP.

**Proof:** To show: $c^T(x - x_0) \geq 0$

$A_{i_1} \ldots A_{i_n}$ are hyperplane defining $x$.

Since $x \in F$: $c^T = \alpha_{i_1} A_{i_1} + \alpha_{i_2} A_{i_2} + \cdots + \alpha_{i_n} A_{i_n}$, $\alpha_{i_1} \ldots \alpha_{i_n} \geq 0$

$$c^T(x - x_0) = \alpha_{i_1} A_{i_1}(x - x_0) + \alpha_{i_2} A_{i_2}(x - x_0) + \cdots + \alpha_{i_n} A_{i_n}(x - x_0)$$

$$A_{i_j}(x) = b_{i_j} \qquad A_{i_j}(x_0) \leq b_{i_j} \qquad \Rightarrow A_{i_j}(x - x_0) \geq 0$$

$$\Rightarrow c^T(x - x_0) \geq 0$$

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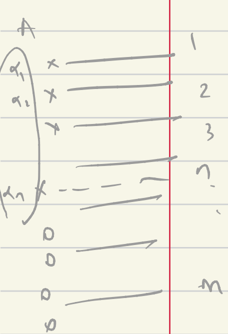
- For $x \in F$, with $A_{i_1} x = b_{i_1}, \ldots A_{i_n} x = b_{i_n}$

suppose $c^T = \alpha_{i_1} A_{i_1} + \alpha_{i_2} A_{i_2} + \cdots + \alpha_{i_n} A_{i_n}$, $\alpha_{i_1} \ldots \alpha_{i_n} \geq 0$

define $y \in \mathbb{R}^m$ : $\langle y_1, y_2, \ldots, y_m \rangle$




$y_{i_1} = \alpha_{i_1}$
$\vdots$
$y_{i_n} = \alpha_{i_n}$

$y_j = 0$ for $j \notin \{\alpha_{i_1} \ldots \alpha_{i_n}\}$

**Then:** $\boxed{c^T x = b^T y}$

define $\bar{F} = \{ \langle y_1, y_2, \ldots y_m \rangle$ as above $|$ for each $x \in F \}$

**Lemma:** $\min \{ b^T y \mid y \in \bar{F} \} = \min \{ c^T x \mid x \in F \} =$ optimum of primal

- For $x \in F$, with $A_{i_1} x = b_{i_1}, \dots \quad A_{i_n} x = b_{i_n}$

suppose $\quad c^T = \alpha_{i_1} A_{i_1} + \alpha_{i_2} A_{i_2} + \dots + \alpha_{i_n} A_{i_n}, \quad \alpha_{i_1} \dots \alpha_{i_n} \geq 0$

define $\quad y \in \mathbb{R}^m : \quad \langle y_1, y_2, \dots, y_m \rangle$

$$y_{i_1} = \alpha_{i_1}$$

$$\vdots \qquad\qquad y_j = 0 \quad \text{for } j \notin \{\alpha_{i_1} \dots \alpha_{i_n}\}$$

$$y_{i_n} = \alpha_{i_n}$$

$$\therefore \quad c^T = y_1 A_1 + y_2 A_2 + \dots + y_n A_n, \qquad y_i \geq 0$$

$$c^T x = y_1 A_1 x + y_2 A_2 x + \dots + y_n A_n x$$

$$= y_{i_1} A_{i_1} x + y_{i_2} A_{i_2} x + \dots + y_{i_n} A_{i_n} x + 0 \dots$$

$$= b^T y.$$

where $\quad x \in F$

## MOTIVATION FOR THE DUAL:

Now consider the LP:

Primal:

maximize $c^T x$

subj to $Ax \leq b$

$Ax \leq b$

$y_1 \times$ $A_1$

$\vdots$

$y_m \times$ $A_m$

$$\boxed{\begin{aligned} &\text{minimize} \quad b^T y \\ &\text{subject to:} \quad A^T y = c \\ &\qquad\qquad\qquad y \geq 0 \end{aligned}}$$

cost is a non-negative combination of hyperplanes

→ Above LP is the dual of our considered primal.

**Theorem:** {Extreme points of dual} equals $\bar{F}$

Assuming this theorem: + fact that $c^T x_0 \leq b^T y$ $\forall y$ feasible (dual)

$$\text{Optimum (dual)} = \min \{ b^T y \mid y \in \bar{F} \} = \text{Optimum (Primal)}$$

- $F = \{ \langle x_1, x_2, \ldots x_n \rangle \in \mathbb{R}^n \mid \exists\, i_1, i_2, \ldots i_n.$

  - $A_{i_1} x = b_{i_1}$

    $\vdots$

    $A_{i_n} x = b_{i_n}$
  - $A_{i_1} \ldots A_{i_n}$ are lin. ind.
  - $c = \alpha_{i_1} A_{i_1} + \cdots + \alpha_{i_n} A_{i_n}$

    $\alpha_{i_1} \ldots \alpha_{i_n} \geq 0 \}$

$\downarrow$

For $x$ given by , we have a $y$ as follows:

$\Rightarrow$ $y_{i_1} = \alpha_{i_1}$ $\quad y_j = 0 \ \forall$ others.

$\vdots$

$y_{i_n} = \alpha_{i_n}$

- $\bar{F} = \{ \langle y_1 \ldots y_m \rangle \mid$ obtained from each $x \in F \}$.

- Optimum of primal $x_0 \in F$. There will be a corresponding $y_0$.

- $\forall$ each $x \in F$, $c^T x = b^T y$ where $y$ corresponds to $x$.

- $\min \{ c^T x \mid x \in F \} = \min \{ b^T y \mid y \in \bar{F} \} = \text{optimum (primal)}$

  follows because $c^T x_0 \leq c^T x$ for all $x \in F$.

**Remains to show:** $\bar{F}$ is the set of extreme points of dual.

- Since dual is in equational form, this is equivalent to showing:

  $\bar{F}$ is the set of all bfs's of dual.

  $\hookrightarrow$ (Exercise)

## Summary:

- Consider all intersection points of primal such that cost is a non-negative combination of the rows corresponding to the defining hyperplanes.
- Optimum (primal) is a point with least cost in the above set.
- Leads to the definition of dual, and duality theorem.