# LINEAR OPTIMIZATION

LECTURE 14

3/6/2021

# Solutions to Problem Set 3:

1.

(a) Maximize $x_1 - 2x_2 + x_3$

Subject to
$$
\begin{aligned}
x_1 + 2x_2 + x_3 &\le 12 \\
2x_1 + x_2 - x_3 &\le 6 \\
-x_1 + 3x_2 &\le 9 \\
x_1,\ x_2,\ x_3 &\ge 0
\end{aligned}
$$

Add slack variables $s_1, s_2, s_3$ to bring to equational form.

$$
\begin{aligned}
s_1 &= 12 - x_1 - 2x_2 - x_3 \\
s_2 &= 6 - 2x_1 - x_2 + x_3 \\
s_3 &= 9 + x_1 - 3x_2 \\
\hline
z &= x_1 - 2x_2 + x_3
\end{aligned}
$$

$$\Big\downarrow \quad x_3 \uparrow \quad s_1 \downarrow$$

$$
\begin{aligned}
x_3 &= 12 - x_1 - 2x_2 - s_1 \\
s_2 &= 18 - 3x_1 - 3x_2 - s_1 \\
s_3 &= 9 + x_1 - 3x_2 \\
\hline
z &= 12 - 4x_2 - s_1
\end{aligned}
$$

Optimum: $x_1 = 0$, $x_2 = 0$, $x_3 = 12$, $s_1 = 0$, $s_2 = 18$, $s_3 = 9$

Cost $= 12$.

1. (b) Maximize $3x_1 + 5x_2$

Subject to
$$\begin{aligned} x_1 - 2x_2 &\le 6 \\ x_1 &\le 10 \\ x_2 &\ge 1 \rightarrow -x_2 \le -1 \\ x_1, x_2 &\ge 0 \end{aligned}$$

$\downarrow$

$$\begin{aligned} x_1 - 2x_2 + s_1 &= 6 \\ x_1 + s_2 &= 10 \\ -x_2 + s_3 &= -1 \\ x_1, x_2, x_3, s_1, s_2, s_3 &\ge 0 \end{aligned}$$

$\{s_1, s_2, s_3\}$ is not a feasible basis.

- To find the initial basis, we can use the method using an auxiliary LP that was discussed in the lecture.

Now we want to guess an initial feasible basis.

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$
$$\quad\quad\uparrow\quad\uparrow\quad\uparrow$$

Consider $\{s_1, s_2, x_2\}$. Columns are linearly independent

Putting $x_1 = 0$, $s_3 = 0$ gives: $x_2 = 1$, $s_1 = 8$, $s_2 = 10$

$\rightarrow$ feasible.

$$\begin{aligned} x_1 - 2x_2 + s_1 &= 6 \\ x_1 + s_2 &= 10 \\ -x_2 + s_3 &= -1 \end{aligned}$$

$$x_1,\ x_2,\ x_3,\ s_1,\ s_2,\ s_3 \geq 0$$

Initial feasible basis: $\{ s_1, s_2, x_2 \}$

$$\begin{aligned} -2x_2 + s_1 &= 6 - x_1 \\ s_2 &= 10 - x_1 \\ -x_2 &= -1 - s_3 \end{aligned}$$

Putting (3) in (1):

$$\begin{aligned} +2 + 2s_3 + s_1 &= 6 - x_1 \\ s_2 &= 10 - x_1 \\ -x_2 &= -1 - s_3 \end{aligned}$$

Rewriting:

$$\boxed{\begin{aligned} s_1 &= 4 - x_1 - 2s_3 \\ s_2 &= 10 - x_1 \\ x_2 &= 1 + s_3 \\ \hline z &= 5 + 3x_1 + 5s_3 \end{aligned}}$$

$$3x_1 + 5x_2 = 3x_1 + 5(1+s_3)$$

↳ Initial tableau.

→ continue simplex from here.

$$
\begin{aligned}
s_1 &= 4 - x_1 - 2s_3 \\
s_2 &= 10 - x_1 \\
x_2 &= 1 + s_3 \\
\hline
z &= 5 + 3x_1 + 5s_3
\end{aligned}
$$

$$\Big\downarrow \; x_1 \uparrow \quad s_1 \downarrow$$

$$
\begin{aligned}
x_1 &= 4 - s_1 - 2s_3 \\
s_2 &= 6 + s_1 + 2s_3 \\
x_2 &= 1 + s_3 \\
\hline
z &= 17 - 3s_1 - s_3
\end{aligned}
$$

Optimum: $x_1 = 4, \; x_2 = 1, \; s_1 = 0, \; s_2 = 6, \; s_3 = 0$

Cost 17

2. Input: a set of inequalities $Ax \leq b$

Output: is $Ax \leq b$ feasible?

- Convert $Ax \leq b$ to equational form.

  - Assume the equational form is: $A'y = b'$
    $y \geq 0$

Consider the auxiliary LP used to find an initial feasible basis for the simplex.

- Add extra variables $s_1, s_2, \ldots s_m$

maximize $-s_1 - s_2 \ldots - s_m$

$$A'_1 y + s_1 = b'_1$$
$$A'_2 y + s_2 = b'_2$$
$$\vdots$$
$$A'_m y + s_m = b'_m$$
$$y \geq 0 \quad \bar{s} \geq 0$$

$Ax \leq b$ is feasible iff above LP has optimum $= 0$

3.

a) Maximize $5x_1 - x_2 + 2x_3$

Subject to:

$$
\begin{aligned}
x_1 - 6x_2 + x_3 &\leq 2 \\
5x_1 + 7x_2 - 2x_3 &\leq -4 \\
8x_1 - 10x_2 + 19x_3 &\leq 75 \\
5x_2 + 14x_3 &\leq 30
\end{aligned}
$$

$\downarrow$ equational form:

Maximize $5x_1^+ - 5x_1^- - x_2^+ + x_2^- + 2x_3^+ - 2x_3^-$

subject to:

$$
\begin{bmatrix}
1 & -1 & -6 & 6 & 1 & -1 \\
5 & -5 & 7 & -7 & -2 & 2 \\
8 & -8 & -10 & 10 & 19 & -19 \\
0 & 0 & 5 & -5 & 14 & -14
\end{bmatrix}
\begin{bmatrix}
x_1^+ \\ x_1^- \\ x_2^+ \\ x_2^- \\ x_3^+ \\ x_3^-
\end{bmatrix}
=
\begin{bmatrix}
2 \\ -4 \\ 75 \\ 30
\end{bmatrix}
$$

$x_1^+, x_1^-, x_2^+, x_2^-, x_3^+, x_3^- \geqslant 0$

b) Maximize $2x_1 - 3x_2 + 4x_3$

subject to: $-x_1 + 2x_2 - x_3 = 14$

$5x_1 - 6x_2 + 12x_3 = 20$

$x_1, \quad x_2, \quad x_3 \geq 0$

$\downarrow$ general form

Maximize $2x_1 - 3x_2 + 4x_3$

subject to:

$$\begin{bmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \\ 5 & -6 & 12 \\ -5 & 6 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 14 \\ -14 \\ 20 \\ -20 \end{bmatrix}$$

$x_1, \ x_2, \ x_3 \geq 0$

3. (c) General form LP is degenerate

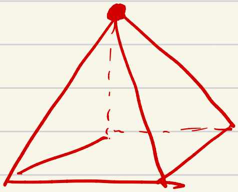
Question CANCELLED

$$Ax \le b$$

Some feasible point has more than 'n' hyperplanes passing through it.

Let us assume that this point satisfies:

$$A_1 x = b_1$$
$$A_2 x = b_2$$
$$\vdots$$
$$A_n x = b_n$$
$$A_{n+1} x = b_{n+1}$$
$$A_{n+2} x < b_{n+2}$$
$$\vdots$$
$$A_m x < b_m$$



(i) $C = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ $D = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{bmatrix}$

Claim : Columns of $C$ are linearly independent.

↳ Not clear!

→ Not sure if question is correct!

3. (d) Equational form LP is degenerate $\Rightarrow$ General form LP is degenerate.

$$\left.\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}\right\} \longrightarrow m \leq n$$

There is a bfs which has multiple bases. $\Big\} \rightarrow x^* \Rightarrow$ there are strictly more than $n-m$ 0's in $x^*$

General form:

$$\left.\begin{aligned} A_1 x &\leq b_1 \\ -A_1 x &\leq -b_1 \\ A_2 x &\leq b_2 \\ -A_2 x &\leq -b_2 \\ &\vdots \\ A_m x &\leq b_m \\ -A_m x &\leq -b_m \end{aligned}\right\} m \text{ hyperplanes}$$

$$\left.\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \\ &\vdots \\ x_n &\geq 0 \end{aligned}\right\} n \text{ hyperplanes}$$

We need to find more than 'n' hyperplanes passing through $x^*$.

- $A_1 x = b_1$ $\qquad$ $x_j = 0 \quad \forall j$ s.t. $x^*(j) = 0$

$\vdots$

$A_1 x = b_m$ $\qquad$ We know that there are $> n-m$ such variables.

$\underbrace{\phantom{xxxx}}_{m \text{ of them.}}$

Therefore, more than $n$ hyperplanes pass through $x^*$.

4. LP: $n$ variables unconstrained. $\{x_1, x_2, \dots x_n\}$

Add variable $\underline{\underline{x_0}}$.

$x_1 - x_0$

Initial LP: $Ax \leq b$

We consider variables: $y_0, y_1, \dots y_n$

Old

$$\boxed{\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots \; a_{mn}x_n \leq b_m \end{array}}$$

$\rightarrow$

New

$$\boxed{\begin{array}{c} a_{11}(y_1 - y_0) + \cdots + a_{1n}(y_n - y_0) \leq b_1 \\ \\ a_{m1}(y_1 - y_0) + \cdots + a_{mn}(y_n - y_0) \leq b_m \end{array}}$$

$y_0, y_1, y_2, \dots y_m \geq 0$

Claim: 1: For every feasible soln. $(x_1, \dots x_n)$ of old LP, there exists a feasible soln. $(y_0, y_1 \dots y_n)$ of new LP.

Suppose min over $\{x_1, \dots x_n\}$ is given by $x_i$.
This means: $x_j - x_i \geq 0 \;\; \forall j$

We want:

$$\begin{aligned} y_1 - y_0 &= x_1 \\ y_2 - y_0 &= x_2 \\ &\vdots \\ y_n - y_0 &= x_n \end{aligned}$$

Moreover:

$y_0 + x_1$

$y_0 + x_2$

$y_0 \vdots x_m \geq 0$

We want:

$$y_1 - y_0 = x_1$$
$$y_2 - y_0 = x_2$$
$$\vdots$$
$$y_n - y_0 = x_n$$

Moreover:

$$y_0 + x_1$$
$$y_0 + x_2$$
$$\vdots$$
$$y_0 + x_m \geq 0$$

Choose $y_0$ to be a "large enough" number.

Suppose the least value among $\{x_1, \dots x_n\}$ is given by $x_i$.

- if $x_i \geq 0 \rightarrow y_0 = 0$,

else, $x_i < 0 \rightarrow y_0 = -x_i$

$\hookrightarrow$ $y_0 \geq 0$

$$y_j = y_0 + x_j = -x_i + x_j \geq 0$$

since $x_i$ was the least.

Claim 2: From every soln. $(y_0, y_1, \dots y_n)$, we get a soln. for old LP $(x_1, \dots x_n)$

$$x_i = y_i - y_0$$

## Alternate solution (based on Anurag's idea)

Initial LP: $Ax \leq b$

In the new LP consider variables $y_1, y_2, \ldots, y_n, y_{n+1}$

| Old LP: | New LP: |
|---|---|
| $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$ | $a_{11}(y_1 - y_2) + a_{12}(y_2 - y_1) + \cdots + a_{1n}(y_n - y_{n+1}) \leq b_1$ |
| $\vdots$ | $\vdots$ |
| $a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$ | $a_{m1}(y_1 - y_2) + a_{m2}(y_2 - y_1) + \cdots + a_{mn}(y_n - y_{n+1}) \leq b_m$ |
| | $y_1, y_2, \ldots, y_n, y_{n+1} \geq 0$ |

**Claim 1:** Every soln. to new LP has equivalent soln. in old LP.

Suppose $\langle y_1^*, y_2^*, \ldots, y_n^*, y_{n+1}^* \rangle$ is a soln. to new LP.

Take $x_i = y_i^* - y_{i+1}^*$ $\forall i \in \{1, \ldots, n\}$.

**Claim 2:** Every soln. in old LP has equivalent soln. in new LP.

Suppose $\langle x_1^*, x_2^*, \ldots, x_{n-1}^*, x_n^* \rangle$ is a soln. to old LP.

Take

$$y_1 = y_{n+1} + x_1^* + x_2^* + \cdots + x_n^*$$
$$y_2 = y_{n+1} + x_2^* + \cdots + x_n^*$$
$$\vdots$$
$$y_i = y_{n+1} + x_i^* + \cdots + x_n^*$$
$$\vdots$$
$$y_n = y_{n+1} + x_n^*$$

$y_{n+1}$ = a large enough, non-negative value so that $y_1 \ldots y_n$ are all non-negative

5.

a) Maximize $2x_1 - 12x_2 + 20x_3$ | Minimize $25y_1 + 15y_2 + 4y_3$

Subj. to $6x_1 + 9x_2 + 25x_3 \leq 25$ → Subj. to $6y_1 + 2y_2 + 4y_3 \geq 2$

$2x_1 - 6x_2 + 3x_3 = 15$ | $9y_1 - 6y_2 + 7y_3 \leq -12$

$4x_1 + 7x_2 - 20x_3 \geq 4$ | $25y_1 + 3y_2 - 20y_3 = 20$

$x_1 \geq 0$ | $y_1 \geq 0$

$x_2 \leq 0$ | $y_2$ unrest.

$x_3$ unrest. | $y_3 \leq 0$

b) Maximize $8x_1 + 3x_2 - 2x_3$ | Minimize $2y_1 - 4y_2$

$x_1 - 6x_2 + x_3 \geq 2$ → $y_1 + 5y_2 \leq 8$

$5x_1 + 7x_2 - 2x_3 = -4$ | $-6y_1 + 7y_2 \geq 3$

$x_1 \leq 0$ | $y_1 - 2y_2 = -2$

$x_2 \geq 0$ | $y_1 \geq 0$

$y_2$ unrest.

c) Minimize $-2x_1 + 3x_2 + 5x_3$ | Maximize $5y_1 + 4y_2 + 4y_3$

$-2x_1 + x_2 + 3x_3 \geq 5$ → $-2y_1 + 2y_2 \leq -2$

$2x_1 + x_3 \leq 4$ | $y_1 + 2y_3 \geq 3$

$2x_2 + x_3 = 4$ | $3y_1 + y_2 + y_3 = 5$

$x_1 \leq 0$ | $y_1 \leq 0$

$x_2 \geq 0$ | $y_2 \geq 0$

$x_3$ unrestr. | $y_3$ unrest.

6. **Primal:**

maximize $x_1$

Subj. to

$$x_1 - x_2 \leq 1$$
$$-x_1 + x_2 \leq -2$$

Dual:

minimize: $y_1 - 2y_2$

$$y_1 - y_2 \geq 1$$
$$-y_1 + y_2 \geq 0$$

7. Primal:

maximize $c^T x$

subject to: $Ax = b$

Dual

minimize $b^T y$

subject to $A^T y = c$

From weak duality: $c^T \bar{x} \leq b^T \bar{y}$ $\forall \bar{x} \in$ primal, $\forall \bar{y} \in$ dual

$$c^T x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$c_1 = a_{11} y_1 + a_{21} y_2 + \cdots + a_{n1} y_n$$

$$\vdots$$

$$c_n = a_{1n} y_1 + a_{2n} y_2 + \cdots + a_{mn} y_n$$

$$c^T x = (a_{11} y_1 + \cdots + a_{n1} y_n) x_1 + \cdots + (a_{1n} y_1 + \cdots + a_{mn} y_n) x_n$$

Regroup in terms of y: $(a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n) y_1 + \cdots + (a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n) y_m$

$$= b_1 y_1 + b_2 y_2 + \cdots + b_m y_m = b^T y.$$