LINEAR OPTIMIZATION

LECTURE 13

PROOF OF DUALITY VIA SIMPLEX

maximize $c^T x$

subject to $Ax \leq b$

$x \geq 0$

PRIMAL (P)

DUAL (D)

minimize $b^T y$

subject to $A^T y \geq c$

$y \geq 0$

WEAK DUALITY:

For every feasible solution $\bar{x}$ of (P),
for every feasible solution $\bar{y}$ of (D):

$$c^T \bar{x} \leq b^T \bar{y}$$

STRONG DUALITY: Exactly one of the foll. occurs:

- 1. Both (P) and (D) infeasible
- 2. (P) unbounded, (D) infeasible
- 3. (P) infeasible, (D) unbounded
- 4. Optimum (P) = x^*, Optimum (D) = y^*

$$\boxed{c^T x^* = b^T y^*}$$

(P) \ (D)	infeasible	unbounded	$\exists$ optimum
infeasible	✓	✓	✗
unbounded	✓	✗	✗
$\exists$ optimum	✗	✗	✓

Exercise:

Write the dual of the following LP:

maximize $5 + 3x_1 + 4x_2 - 2x_3$

subject to
$$x_1 - 7x_2 + 5x_3 \leq 10$$
$$2x_1 + 3x_2 - x_3 \leq 15$$
$$x_2 + 7x_3 \leq 20$$
$$x_1, x_2, x_3 \geq 0$$

Dual:

minimize $5 + 10y_1 + 15y_2 + 20y_3$

subject to:
$$y_1 + 2y_2 \geq 3$$
$$-7y_1 + 3y_2 + y_3 \geq 4$$
$$5y_1 - y_2 + 7y_3 \geq -2$$
$$y_1, y_2, y_3 \geq 0$$

GOAL: Proof of duality theorem, using simplex

REFERENCE: Section 6.3 of text:

Understanding and Using Linear Programming
— Matoušek & Gärtner

We will prove the following:

> When primal has an optimum,
> - the dual is feasible and
> - optimum of dual coincides with optimum of primal

STEP 1: Consider primal to be in equational form *

max	$c^T x$
subject to	$Ax = b$
	$x \geq 0$

Primal

minimize	$b^T y$
subject to	$A^T y \geq c$
	y unrestricted

Dual

$$\begin{array}{ccc} \text{Primal} & \text{---} & \text{Dual} \\ \text{elem. opn} \; \Big| & & \Big| = \text{optima} \\ \text{Primal}' & \text{---} & \text{Dual}' \end{array}$$

STEP 2: Elementary operations preserve the dual optimum

$\max \; c_1x_1 + c_2x_2 + \cdots + c_nx_n$	$\min \; b_1y_1 + b_2y_2 + \cdots + b_my_m$
$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$	$a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1$
$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$	$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq c_2$
$\vdots$	$\vdots$
$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$	$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n$
$x_1, x_2, \ldots, x_n \geq 0$	$y_1, y_2, \ldots y_m$ unrestricted

Multiplying a primal equation by a scalar.

$$\vdots$$
$$\alpha a_{i1}x_1 + \alpha a_{i2}x_2 + \cdots + \alpha a_{in}x_n = \alpha b_i$$
$$\vdots$$

Set of solutions does not change

$$\min \; \cdots + \alpha b_i y_i + \cdots$$
$$+ \alpha a_{i1} y_i +$$
$$\cdots + \alpha a_{i2} y_i + \cdots$$
$$\vdots$$
$$+ \alpha a_{in} y_i + \cdots$$

$$(y_1', y_2', \ldots, y_m') \leftrightarrow \left(y_1', \ldots, \frac{y_i'}{\alpha}, \ldots, y_m'\right)$$

Solution to original dual — Solution to modified dual

— costs are same —

Replacing a primal equation i by sum of equations i and j (Exercise).

Replacing a primal equation i by sum of equations i and j

max $c_1x_1 + c_2x_2 + \dots + c_nx_n$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$x_1, x_2, \dots, x_n \geq 0$

min $b_1y_1 + b_2y_2 + \dots + b_my_m$

D1

$$a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m \geq c_1$$
$$a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m \geq c_2$$
$$\vdots$$
$$a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m \geq c_n$$

$y_1, y_2, \dots y_m$ unrestricted

$$\vdots$$

$$(a_{i1} + a_{j1})x_1 + \dots + (a_{in} + a_{jn})x_n = b_i + b_j$$

min $\dots (b_i + b_j)y_i \dots$

D2

$$\dots + (a_{i1} + a_{j1})y_i + \dots$$
$$\dots + (a_{i2} + a_{j2})y_i + \dots$$
$$\vdots$$
$$\dots + (a_{in} + a_{jn})y_i + \dots$$

If $y_1 = y_1'$, $y_2 = y_2'$, $\vdots$, $y_m = y_m'$ is a soln. of D1

then $y_1 = y_1'$, $\vdots$, $y_j = y_j' - y_i'$, $\vdots$, $y_i = y_i'$, $\vdots$, $y_m = y_m'$ is a soln. of D2 with same cost.

Conversely:

If $y_1 = y_1''$

$\vdots$

$y_m = y_m''$

is a soln. of D2

then $y_1 = y_1''$

$\vdots$

$y_j = y_i'' + y_j''$

$\vdots$

$y_i = y_i''$

$\vdots$

$y_m = y_m''$

is a soln. of D1 with same cost.

STEP 3. Observe that simplex tableaus are obtained through a sequence of elementary operations starting from the original system of equations

$$Ax = b \qquad x \geq 0$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\Big\downarrow \text{ elementary operations}$$

Basis

$$\begin{array}{c} \\ i_1 \\ i_2 \\ \vdots \\ i_m \end{array} \begin{array}{c} \begin{array}{cccc} i_1 & i_2 & \cdots & i_m \end{array} \\ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ 0 & 0 & & \vdots \\ 0 & \vdots & & \\ 0 & \vdots & & 1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1' \\ b_2' \\ \vdots \\ b_m' \end{bmatrix}$$

$$\Big\downarrow$$

$$x_B = p + Q x_N$$

$$z = z_0 + r^T x_N$$

non-basic

$$x_{i_1} = p_1 + \boxed{\cdots}$$

$$x_{i_2} = p_2 + \boxed{}$$

$$\vdots$$

$$x_{i_m} = p_m + \boxed{}$$

From Step 2 and Step 3: Dual optimum of original system is the same as dual optimum of a simplex tableau corresponding to original system

Step 4: What is the dual when primal is given by a simplex tableau? *

$x_{i_1} = p_1 + q_{11} x_{j_1} + \dots + q_{1n-m} x_{j_{n-m}}$

$\vdots$

$$x_B = p + Q x_N$$

$$B = \{ i_1, i_2, \dots i_m \} \quad N = \{ j_1, j_2, \dots j_{n-m} \}$$

$$Q \quad \begin{bmatrix} q_{11} & q_{12} & \dots & q_{1,n-m} \\ \vdots & & & \\ q_{m1} & \dots & \dots & q_{m,n-m} \end{bmatrix} \begin{matrix} x_{j_1} \\ \vdots \\ \vdots \\ x_{j_{n-m}} \end{matrix}$$

Equations:

$$x_{i_1} - q_{11} x_{j_1} - q_{12} x_{j_2} - \dots - q_{1(n-m)} x_{j_{n-m}} = p_1$$

$$x_{i_2} - q_{21} x_{j_1} - q_{22} x_{j_2} - \dots - q_{2(n-m)} x_{j_{n-m}} = p_2$$

$$\vdots$$

$$x_{i_m} - q_{m1} x_{j_1} - q_{m2} x_{j_2} - \dots - q_{m(n-m)} x_{j_{n-m}} = p_m$$

Cost: $c_1 x_1 + c_2 x_2 + \dots + c_n x_n = c_{i_1} x_{i_1} + c_{i_2} x_{i_2} + \dots + c_{i_m} x_{i_m} + c_{j_1} x_{j_1} + \dots + c_{j_{n-m}} x_{j_{n-m}}$

$\hookrightarrow$ can be rewritten using $j_1 \dots j_{n-m}$

$$\boxed{= \underbrace{(c_{i_1} p_1 + c_{i_2} p_2 + \dots + c_{i_m} p_m)}_{z_0} + \underbrace{(c_{j_1} + c_{i_1} q_{11} + c_{i_2} q_{21} + \dots + c_{i_m} q_{m1}) x_{j_1} + \dots + (c_{j_{n-m}} + c_{i_1} q_{1(n-m)} + \dots + c_{i_m} q_{m(n-m)}) x_{j_{n-m}}}_{r^T x_N}}$$

Dual constraints:

$$y_1, y_2, \dots, y_m \geq 0$$

$$-q_{11} y_1 - q_{21} y_2 \dots - q_{m1} y_m \geq \underbrace{c_{j_1} + c_{i_1} q_{11} + c_{i_2} q_{21} + \dots + c_{i_m} q_{m1}}_{r_1}$$

$$\vdots$$

$$-q_{1(n-m)} y_1 - \dots - q_{m(n-m)} y_m \geq \underbrace{c_{j_{n-m}} + c_{i_1} q_{1(n-m)} + \dots + c_{i_m} q_{m(n-m)}}_{r_{n-m}}$$

Equations:

$$
\begin{aligned}
x_{i_1} - q_{11}x_{j_1} - q_{12}x_{j_2} - \cdots - q_{1(n-m)}x_{j_{n-m}} &= p_1 \\
x_{i_2} - q_{21}x_{j_1} - q_{22}x_{j_2} - \cdots - q_{2(n-m)}x_{j_{n-m}} &= p_2 \\
&\vdots \\
x_{i_m} - q_{m1}x_{j_1} - q_{m2}x_{j_2} - \cdots - q_{m(n-m)}x_{j_{n-m}} &= p_m
\end{aligned}
$$

Cost: $c_1x_1 + c_2x_2 + \cdots + c_nx_n = c_{i_1}x_{i_1} + c_{i_2}x_{i_2} + \cdots + c_{i_m}x_{i_m} + c_{j_1}x_{j_1} + \cdots + c_{j_{n-m}}x_{j_{n-m}}$

↳ can be rewritten using $j_1 \cdots j_{n-m}$

$$
\boxed{
\begin{aligned}
= \underbrace{(c_{i_1}p_1 + c_{i_2}p_2 + \cdots + c_{i_m}p_m)}_{z_0} &+ \underbrace{(c_{j_1} + c_{i_1}q_{11} + c_{i_2}q_{21} + \cdots + c_{i_m}q_{m1})x_{j_1} \\
&+ \cdots \\
&+ (c_{j_{n-m}} + c_{i_1}q_{1(n-m)} + \cdots + c_{i_m}q_{m(n-m)})x_{j_{n-m}}}_{r^T x_N}
\end{aligned}
}
$$

Dual constraints:

$$y_1, y_2, \ldots, y_m \geq 0$$

$$-q_{11}y_1 - q_{21}y_2 \cdots - q_{m1}y_m \geq c_{j_1} + c_{i_1}q_{11} + c_{i_2}q_{21} + \cdots + c_{i_m}q_{m1}$$

$$\vdots$$

$$-q_{1(n-m)}y_1 - \cdots - q_{m(n-m)}y_m \geq c_{j_{n-m}} + c_{i_1}q_{1(n-m)} + \cdots + c_{i_m}q_{m(n-m)}$$

STEP 5: Main observation: Assume primal has optimum.

In the optimal tableau, coefficients $r_1, r_2, \ldots r_{n-m}$ are negative!

Hence $y_1, y_2, \ldots, y_m = 0$ is a feasible solution to the dual constraints.

Dual cost: $z_0 + p_1y_1 + p_2y_2 + \cdots + p_my_m$

$= z_0$ when $y_1, y_2, \ldots, y_m = 0$

- Primal optimum $= z_0$ (comes from the final tableau)

- From weak duality, for every x which is a feasible soln. of primal and for every y which is a feasible soln. of dual

  we have $c^T x \leq b^T y$

Now, we know that there is an optimal x^* for primal from simplex tableau, with cost z_0

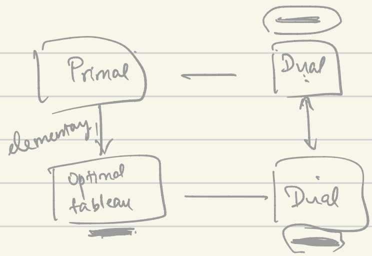
$\therefore$ For every feasible y of dual,

$$z_0 \leq \text{cost}(y)$$

- But we have exhibited one feasible soln. of dual with cost z_0

$\therefore$ optimum (dual) $= z_0$

What does this show?



When primal has an optimum:

- consider the final tableau
- Write the dual constraints for this system
- dual optimum equals the optimum of this system corresponding to final tableau

But, we have seen that elementary operations preserve the dual optimum.

We can also reconstruct the dual feasible solution giving the optimum for the original system.

- Proves duality theorem!