LINEAR OPTIMIZATION

LECTURE 12

27/05/2021

DUAL OF AN LP

GOAL: Given an LP, associate another LP to it (called its "dual") which has interesting relations to the original LP

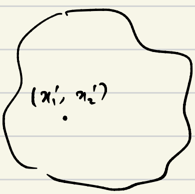
REFERENCE: Section 6.1 of text:

Understanding and Using Linear Programming
– Matoušek & Gärtner

Intuition:

$$\text{maximize} \quad 2x_1 + 3x_2$$

$$\begin{aligned} \text{subject to} \quad 4x_1 + 8x_2 &\le 12 \\ 2x_1 + x_2 &\le 3 \\ 3x_1 + 2x_2 &\le 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$



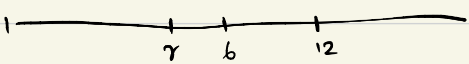
$2x_1' + 3x_2'$

$2x_1' \le 4x_1'$ (since $x_1' \ge 0$)

$3x_2' \le 8x_2'$ (since $x_2' \ge 0$)

$2x_1' + 3x_2' \le 4x_1' + 8x_2' \le 12$ (from LP constraints)

Suppose γ is the optimum of the above LP. $2x_1' + 3x_2 \le 2x_1 + 4x_2$



$\therefore 2x_1 + 4x_2 \le 6$

So the cost: $2x_1 + 3x_2 \le 6$

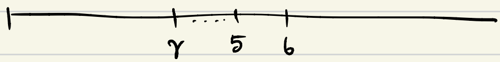
maximize $2x_1 + 3x_2$

subject to

$$\begin{aligned} 4x_1 + 8x_2 &\le 12 \\ 2x_1 + x_2 &\le 3 \\ 3x_1 + 2x_2 &\le 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

add: $\frac{1}{3}(6x_1 + 9x_2) \le 15$

$2x_1 + 3x_2 \le 5$



non negative value

$$\begin{aligned} y_1 \times \quad 4x_1 + 8x_2 &\le 12 \\ y_2 \times \quad 2x_1 + x_2 &\le 3 \\ y_3 \times \quad 3x_1 + 2x_2 &\le 4 \end{aligned}$$

$$(4y_1 + 2y_2 + 3y_3)x_1 + (8y_1 + y_2 + 2y_3)x_2 \le 12y_1 + 3y_2 + 4y_3$$

≥ 2 $\qquad$ ≥ 3

$2x_1 + 3x_2 \le \quad \le$

Minimize $12y_1 + 3y_2 + 4y_3$

$$\begin{aligned} 4y_1 + 2y_2 + 3y_3 &\ge 2 \\ 8y_1 + y_2 + 2y_3 &\ge 3 \\ y_1, y_2, y_3 &\ge 0 \end{aligned}$$

$\underbrace{\gamma \le 12y_1 + 3y_2 + 4y_3}$

Dual of the LP that we started with

optimum (dual) $\ge$ optimum (original LP) non negative value

$$\left\{\begin{array}{lllll} y_1 \times & 4x_1 & + & 8x_2 & \leq 12 \\ y_2 \times & 2x_1 & + & x_2 & \leq 3 \\ y_3 \times & 3x_1 & + & 2x_2 & \leq 4 \end{array}\right.$$

$$\underbrace{(4y_1 + 2y_2 + 3y_3)}_{\geq 2} x_1 + (8y_1 + y_2 + 2y_3) x_2 \leq 12y_1 + 3y_2 + 4y_3$$

If $4y_1 + 2y_2 + 3y_3 \geq 2$ and $8y_1 + y_2 + 2y_3 \geq 3$.

Then. $2x_1 + 3x_2 \leq (4y_1 + 2y_2 + 3y_3)x_1 + (8y_1 + y_2 + 2y_3)x_2$

$\leq 12y_1 + 3y_2 + 4y_3$

for every feasible soln. (x_1, x_2) of original LP.

$\Rightarrow$ optimum of LP $\leq 12y_1 + 3y_2 + 4y_3$ for all (y_1, y_2, y_3) satisfying above constraints

Another LP to minimize this upper bound $12y_1 + 3y_2 + 4y_3$:

Minimize $12y_1 + 3y_2 + 4y_3$

$4y_1 + 2y_2 + 3y_3 \geq 2$

$8y_1 + y_2 + 2y_3 \geq 3$

$y_1, y_2, y_3 \geq 0$

$\rightarrow$ Dual of the LP that we started with

$\rightarrow$ optimum (dual) $\geq$ optimum (original LP) maximize $c^T x$

subject to $Ax \leq b$

$x \geq 0$

Primal

minimize $b^T y$

subject to $A^T y \geq c$

$y \geq 0$

Dual

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$\begin{array}{c} y_1 \times \\ y_2 \times \\ \vdots \\ y_m \times \end{array} \begin{bmatrix} - \; - \; - \; - \; - \\ \\ A : m \times n \\ \\ \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \leq \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$\downarrow$

Objective: Minimize $b^T y$

Constraints: $A^T y \geq c$

$y \geq 0$

Exercise: Write the dual for the following LPs.

1) maximize: $12x_1 + 3x_2 + 5x_3$

$$
\begin{aligned}
7x_1 + 4x_2 + 2x_3 &\leq 100 \\
2x_1 \qquad\quad + 7x_3 &\leq 50 \\
x_1,\ x_2,\ x_3 &\geq 0
\end{aligned}
$$

2) maximize $2x_1 + 3x_2$

subject to:

$$
\begin{aligned}
4x_1 + 8x_2 &\geq 12 \\
2x_1 + x_2 &\leq 3 \\
3x_1 + 2x_2 &\leq 4 \\
x_1,\ x_2 &\geq 0
\end{aligned}
$$

3) minimize $12y_1 + 3y_2 + 4y_3$

subject to

$$
\begin{aligned}
4y_1 + 2y_2 + 3y_3 &\geq 2 \\
8y_1 + y_2 + 2y_3 &\geq 3 \\
y_1,\ y_2,\ y_3 &\geq 0
\end{aligned}
$$

1) maximize: $12x_1 + 3x_2 + 5x_3$

$$7x_1 + 4x_2 + 2x_3 \leq 100$$
$$2x_1 + 7x_3 \leq 50$$
$$x_1, x_2, x_3 \geq 0$$

$\downarrow$

minimize $100y_1 + 50y_2$

$$7y_1 + 2y_2 \geq 12$$
$$4y_1 \geq 3$$
$$2y_1 + 7y_2 \geq 5$$

---

2) maximize $2x_1 + 3x_2$

subject to:

$$y_1 \times \quad 4x_1 + 8x_2 \geq 12$$
$$y_2 \times \quad 2x_1 + x_2 \leq 3$$
$$y_3 \times \quad 3x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

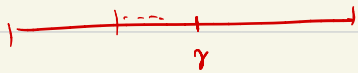
minimize $12y_1 + 3y_2 + 4y_3$

$$4y_1 + 2y_2 + 3y_3 \geq 2$$
$$8y_1 + y_2 + 2y_3 \geq 3 \qquad y_1 \leq 0, \; y_2 \geq 0$$

3)

minimize $\quad 12y_1 + 3y_2 + 4y_3$

subject to
$$x_1 \times \quad 4y_1 + 2y_2 + 3y_3 \geq 2$$
$$x_2 \times \quad 8y_1 + y_2 + 2y_3 \geq 3$$
$$y_1,\ y_2,\ y_3 \geq 0$$



$$(4x_1 + 8x_2)\,y_1 + (2x_1 + x_2)\,y_2 + (3x_1 + 2x_2)\,y_3 \geq 2x_1 + 3x_2$$

Suppose:
$$4x_1 + 8x_2 \leq 12$$
$$2x_1 + x_2 \leq 3$$
$$3x_1 + 2x_2 \leq 4 \qquad x_1,\ x_2,\ x_3 \geq 0$$

Then.
$$12y_1 + 3y_2 + 4y_3 \geq (4x_1 + 8x_2)\,y_1 + (2x_1 + x_2)\,y_2 + (3x_1 + 2x_2)\,y_3$$
$$\geq 2x_1 + 3x_2$$

Dual: maximize $\quad 2x_1 + 3x_2$

subject to:
$$4x_1 + 8x_2 \leq 12$$
$$2x_1 + x_2 \leq 3$$
$$3x_1 + 2x_2 \leq 4$$
$$x_1,\ x_2,\ x_3 \geq 0$$

Primal-dual pairs:

$\max \; c^T x$	$\min \; b^T y$
subj. to $\; Ax \leq b$	subj. to $\; A^T y \geq c$
$x \geq 0$	$y \geq 0$

PRIMAL - DUAL Pairs

Dual of dual is primal.

- Previous example illustrates this

Weak Duality: (Proposition 6.1.1 in the text)

Primal

max. $c^T x$
subj to
$Ax \leq b$
$x \geq 0$

Dual

min. $b^T y$
subj to
$A^T y \geq c$
$y \geq 0$

$c^T x' \leq b^T y'$

x' $\quad$ y'

For every x', y' s.t. x' is a feasible soln. of primal and y' is a feasible soln. of dual.

Corollary:

- If primal is unbounded, dual is infeasible.

Duality Theorem:

Primal opt.

r

= dual optimum

Exactly one of the foll. possibilities occur:

- 1. Both Primal (P) and Dual (D) are infeasible
- 2. Primal is unbounded, dual is infeasible
- 3. (P) is infeasible, dual is unbounded
- 4. Both primal and dual have an optimum. In this case, both the optima coincide.

Writing the dual for different LP forms:

maximize $2x_1 + 3x_2$

subj. to

$y_1 \times \quad 4x_1 + 8x_2 \leq 12$

$y_2 \times \quad 2x_1 + x_2 \leq 3$

$y_3 \times \quad 3x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$

minimize $12y_1 + 3y_2 + 4y_3$

$4y_1 + 2y_2 + 3y_3 \geq 2$

$8y_1 + y_2 + 2y_3 \geq 3$

$y_1, y_2, y_3 \geq 0$

Negative variables:

maximize $2x_1 + 3x_2$

subj. to

$4x_1 + 8x_2 \leq 12 \quad \times y_1$

$2x_1 + x_2 \leq 3 \quad \times y_2$

$3x_1 + 2x_2 \leq 4 \quad \times y_3$

$x_1 \leq 0$

$x_2 \geq 0$

$$(4y_1 + 2y_2 + 3y_3)\,x_1 + (8y_1 + y_2 + 2y_3)\,x_2 \leq 12y_1 + 3y_2 + 4y_3$$

$$2x_1 \leq (4y_1 + 2y_2 + 3y_3)\,x_1$$

Since $x_1 \leq 0$, above inequality is ensured by: $4y_1 + 2y_2 + 3y_3 \leq 2$

maximize $2x_1 + 3x_2$

subj. to

$$4x_1 + 8x_2 \leq 12 \quad \times y_1$$
$$2x_1 + x_2 \leq 3 \quad \times y_2$$
$$3x_1 + 2x_2 \leq 4 \quad \times y_3$$
$$x_1 \leq 0$$
$$x_2 \geq 0$$

$\downarrow$

minimize $12y_1 + 3y_2 + 4y_3$

subject to:

$$4y_1 + 2y_2 + 3y_3 \leq 2$$

$$8y_1 + y_2 + 2y_3 \geq 3$$

$$y_1, y_2, y_3 \geq 0$$

Unconstrained variable:

$$\text{maximize} \quad 2x_1 + 3x_2$$

subj. to

$$4x_1 + 8x_2 \leq 12$$
$$2x_1 + x_2 \leq 3$$
$$3x_1 + 2x_2 \leq 4$$
$$x_2 \geq 0$$

x_1 unconstrained

$$x_1 = x_1^+ - x_1^- \quad , \quad x_1^+, x_1^- \geq 0$$

obj: max : $2x_1^+ - 2x_1^- + 3x_2$

$$4x_1^+ - 4x_1^- + 8x_2 \leq 12$$
$$2x_1^+ - 2x_1^- + x_2 \leq 3$$
$$3x_1^+ - 3x_1^- + 2x_2 \leq 4$$
$$x_2 \geq 0 \qquad x_1^+ \geq 0, \; x_1^- \geq 0$$

$\downarrow$

$$4y_1 + 2y_2 + 3y_3 \geq 2$$
$$-4y_1 - 2y_2 - 3y_3 \geq -2 \quad \rightarrow \quad 4y_1 + 2y_2 + 3y_3 \leq 2$$
$$8y_1 + y_2 + 2y_3 \geq 3$$

Final dual: minimize $12y_1 + 3y_2 + 4y_3$

subject to:

$$4y_1 + 2y_2 + 3y_3 = 2$$
$$8y_1 + y_2 + 2y_3 \geq 3$$
$$y_1, y_2, y_3 \geq 0$$

Constraint with $\geq$:

maximize $2x_1 + 3x_2$

subj. to

$$4x_1 + 8x_2 \geq 12$$
$$2x_1 + x_2 \leq 3$$
$$3x_1 + 2x_2 \leq 4$$
$$x_1 , x_2 \geq 0$$

$$\downarrow$$

minimize $12y_1 + 3y_2 + 4y_3$

subj. to:

$$4y_1 + 2y_2 + 3y_3 \geq 2$$
$$8y_1 + y_2 + 2y_3 \geq 3$$

$$y_1 \leq 0, \quad y_2 \geq 0$$

Constraint with = :

maximize $2x_1 + 3x_2$

subj. to
$$4x_1 + 8x_2 = 12$$
$$2x_1 + x_2 \leq 3$$
$$3x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

$\downarrow$

$$4x_1 + 8x_2 \leq 12 \quad \times y_1^+$$
$$-4x_1 - 8x_2 \leq -12 \quad \times y_1^-$$
$$2x_1 + x_2 \leq 3 \quad \times y_2$$
$$3x_1 + 2x_2 \leq 4 \quad \times y_3$$

$\downarrow$

$$4(y_1^+ - y_1^-) + 2y_2 + 3y_3 \geq 2$$
$$8(y_1^+ - y_1^-) + y_2 + 2y_3 \geq 3$$

$$y_1^+, y_1^-, y_2, y_3 \geq 0$$

$\downarrow$

minimize $12y_1 + 3y_2 + 4y_3$
$$4y_1 + 2y_2 + 3y_3 \geq 2$$
$$8y_1 + 2y_2 + 2y_3 \geq 3$$

y_1 unconstrained, $y_2 \geq 0$, $y_3 \geq 0$

$$\begin{array}{ll} \max & c^T x \\ \text{subj. to} & Ax \le b \\ & x \ge 0 \end{array}$$

$A_i x \ge b_i$ — $y_i \le 0$

$A_i x = b_i$ — y_i is unconstrained

$x_j \le 0$ — $A_j^T y \le c_j$

x_j unconstr. — $A_j^T y = c_j$

$$\begin{array}{ll} \min & b^T y \\ \text{subj. to} & A^T y \ge c \\ & y \ge 0 \end{array}$$

$$\begin{array}{ll} \max & 8x_1 + 3x_2 - 2x_3 \\ \text{subj. to:} & x_1 - 6x_2 + x_3 \ge 2 \\ & 5x_1 + 7x_2 - 2x_3 = -4 \\ & x_1 \le 0,\ x_2 \ge 0,\ x_3 \text{ unrestricted} \end{array}$$

$$\begin{array}{ll} \text{minimize} & 2y_1 - 4y_2 \\ & y_1 + 5y_2 \le 8 \\ & -6y_1 + 7y_2 \ge 3 \\ & y_3 - 2y_2 = -2 \\ & y_1 \le 0 \\ & y_2 \text{ unconstrained} \end{array}$$

Summary:

- Notion of dual; Writing duals for different LP forms
- Weak duality theorem and its proof.
- Statement of strong duality theorem.