LINEAR OPTIMIZATION

LECTURE 11

25/05/2021

Lecture 11: Bland's rule

In the last lecture, we have shown that when the simplex method terminates, it does so with a correct answer.

But: does Simplex terminate always?

– No.

We have seen an example previously where the pivoting did not change the bfs. This happens when several bases correspond to a single bfs.

This degenerate pivoting step may potentially lead to cycling.

We will now see a pivoting rule that prevents cycling.

Bland's rule:

Assume variables are $\{x_1, x_2, \dots, x_n\}$

Tableau B

x_{k_1}	$p_1 + q_{11} x_{l_1} + \cdots +$
x_{k_2}	p_2
$\vdots$	$\vdots$
x_{k_m}	p_m
z	$z_0 + r_1 x_{l_1} + \cdots$

basic var. B

non-basic vars. N

Choice of entering / leaving variable: If there are several possibilities choose the variable with the smallest index

Example:

Maximize $\quad x_1 - 2x_2 + x_3$

Subject to:

$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 12 \\ 2x_1 + x_2 - x_3 &\leq 6 \\ -x_1 + 3x_2 &\leq 9 \\ x_1,\ x_2,\ x_3 &\geq 0 \end{aligned}$$

- Add slack variables x_4, x_5, x_6.

$$\begin{aligned} x_4 &= 12 - x_1 - 2x_2 - x_3 \\ x_5 &= 6 - 2x_1 - x_2 + x_3 \\ x_6 &= 9 + x_1 - 3x_2 \\ \hline z &= x_1 - 2x_2 + x_3 \end{aligned}$$

$$\Big\downarrow \quad \begin{matrix} x_1 \uparrow \\ x_5 \downarrow \end{matrix}$$

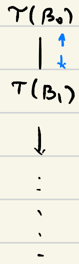
$$\begin{aligned} x_4 &= 9 + \tfrac{1}{2}x_5 - \tfrac{3}{2}x_2 - \tfrac{3}{2}x_3 \\ x_1 &= 3 - \tfrac{1}{2}x_5 - \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 \\ x_6 &= 12 - \tfrac{1}{2}x_5 - \tfrac{7}{2}x_2 \\ \hline z &= 3 - \tfrac{1}{2}x_5 - \tfrac{5}{2}x_2 + \tfrac{3}{2}x_3 \end{aligned}$$

$$\Big\downarrow \quad \begin{matrix} x_3 \uparrow \\ x_4 \downarrow \end{matrix}$$

. . . . . .

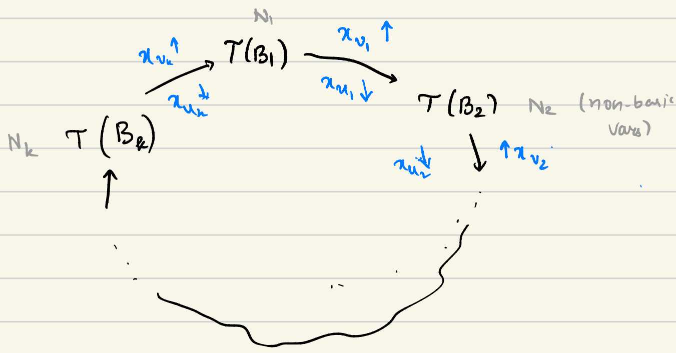
Theorem: The simplex method with Bland's pivoting rule terminates.

Proof:



Suppose simplex with Bland's rule for the entering and leaving variables does not terminate.

There is a cycle of tableaus:

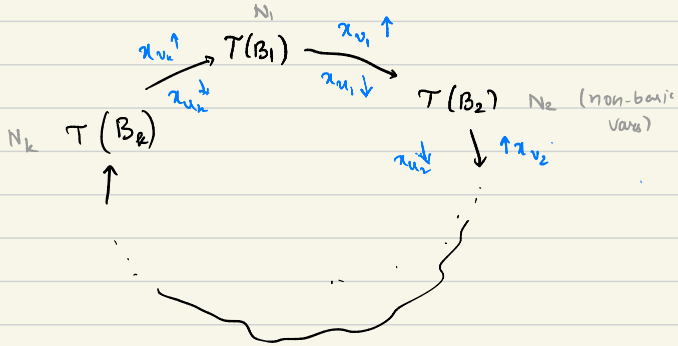


Claim 1: $\{x_{v_1}, x_{v_2}, \ldots x_{v_k}\} = \{x_{u_1}, x_{u_2}, \ldots, x_{u_k}\}$

x_{u_i} leaves the basis at B_i $\qquad B_i \xrightarrow[x_{u_i} \downarrow]{}$

In the path from $B_{i+1} \rightsquigarrow B_i$ back to B_i, somewhere u_i has entered the basis again, because $x_{u_i} \in B_i$.

$\therefore x_{u_i} \in \{x_{v_1}, x_{v_2} \ldots x_{v_k}\}$



Claim 1: $\{x_{v_1}, x_{v_2}, \dots x_{v_k}\} = \{x_{u_1}, x_{u_2}, \dots, x_{u_k}\}$

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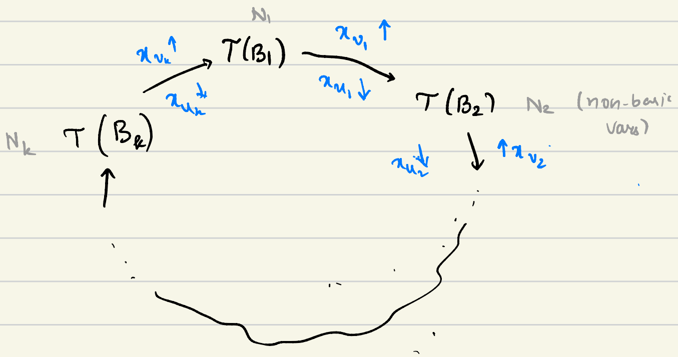
$\therefore \; x_{u_i} \in \{x_{v_1}, x_{v_2} \dots x_{v_k}\}$

$$\{x_{u_1}, x_{u_2} \dots x_{u_k}\} \subseteq \{x_{v_1}, x_{v_2} \dots x_{v_k}\}$$

Similar for x_v's. $\{x_{v_1}, x_{v_2}, \dots x_{v_k}\} \subseteq \{x_{u_1}, \dots x_{u_k}\}$

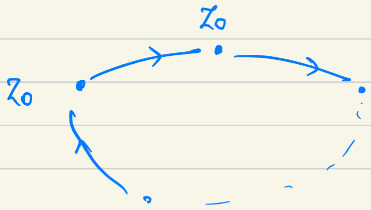
$\therefore$ Let us call $F = \{x_{v_1}, x_{v_2}, \dots, x_{v_k}\} = \{x_{u_1}, x_{u_2}, \dots, x_{u_k}\}$

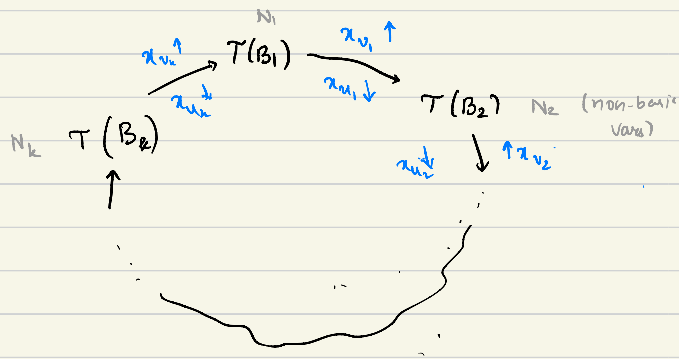


<u>Claim 2:</u> The cost is the same in every tableau in the cycle.

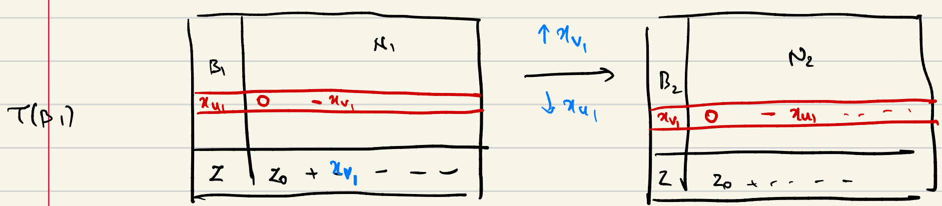
At each pivot step the cost can be greater than or equal to current cost.

$\therefore$ A strict increase of cost is not possible inside the cycle.





<u>Claim 3:</u> The bfs is the same for all the tableaus in the cycle.



Consider x_{v_1} which enters basis B_2.

The tightest constraint is given by x_{u_1}: The constraint should be of the form:

$$x_{u_1} = 0 + \cdots - \alpha x_{v_1} + \cdots \qquad \alpha > 0$$

After rewriting, we have the following constraint on B_2:

$$x_{v_1} = 0 + \cdots - \frac{1}{\alpha} x_{u_1} + \cdots$$

When $x_{u_1} = 0$, we also have $x_{v_1} = 0$

Any other equation would look like this:

In B_1

$$x_e = p_e + \dots + \beta x_{v_1} + \dots$$

In B_2:

$$x_e = p_e + \dots + \beta\left(0 + \dots - \frac{1}{\alpha} x_{u_1} + \dots\right) + \dots$$

When all variables in N_2 are made 0, the quantity:

$$0 + \dots - \frac{1}{\alpha} x_{u_1} + \dots$$

remains 0.

$\therefore$ x_e has same value in B_2 as well.

Claim 4: In this bfs which is the same in all the tableaus of the cycle,

variables in F have value 0.

$$x_F = 0$$

$F = \{ x_{v_1}, x_{v_2}, \dots x_{v_k} \} = \{ x_{u_1}, x_{u_2}, \dots, x_{u_k} \}$

Consider x_{v_i} $\qquad B_i \xrightarrow[\uparrow x_{v_i}]{} B_{i+1}$

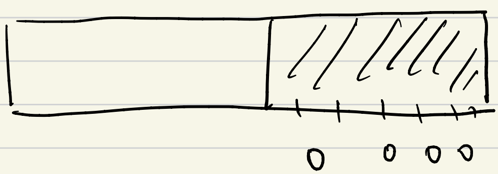
$x_{v_i} \in N_i$

$\therefore$ In the bfs of B_i, $x_{v_i} = 0$

But we the same bfs every where. $\therefore$ $x_{v_i} = 0$ in this bfs, for all x_{v_i}.

What we have: $\{ x_1, x_2, \dots x_n \}$.

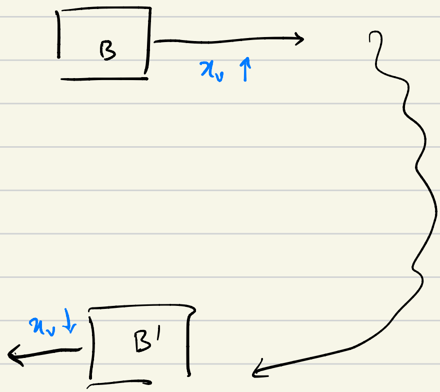
$F \subseteq \{x_1, x_2 \dots x_n\}$



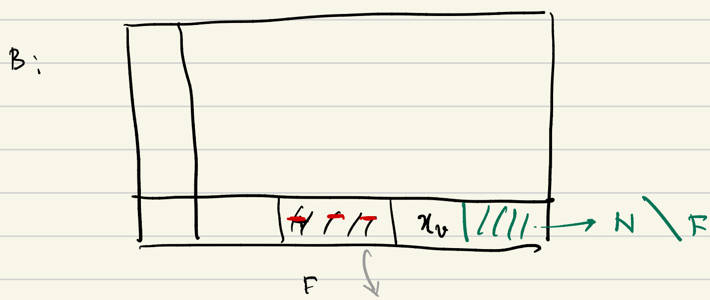
In the bfs of the cycle.

Consider the variable $x_v \in F$ with the highest index.

Suppose x_v enters the basis at B, and x_v leaves at B'



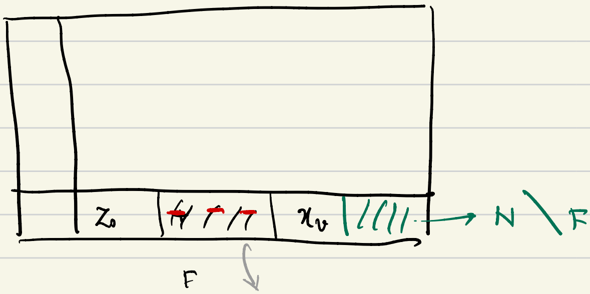
Let us take a closer look at B and B'.



Since we choose x_v over other variables in F, and x_v has highest index, Bland's rule should imply that :

the coefficient of $x_{v'} \neq x_v$, $x_{v'} \in F$, should be negative in the cost row.

B:



Consider the foll. auxiliary LP:

maximize $c^T x$

subject to: $Ax = b$

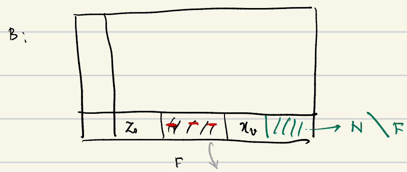
$x_{N \setminus F} = 0$

$x_v \leq 0$

$x_{F \setminus v} \geq 0$

Tableau B corresponds to the same set of equations $Ax = b$.

$\therefore$ The cost function which is written in terms of variables N is the actual cost fn. $c^T x$.



Consider the foll. auxiliary LP:

$$\begin{aligned} &\text{maximize} && c^T x \\ &\text{subject to:} && Ax = b \\ &&& x_{N\setminus F} = 0 \\ &&& x_v \leq 0 \\ &&& x_{F\setminus v} \geq 0 \end{aligned}$$

<u>Claim</u>: The aux. LP is bounded by z_0.

Because: $\text{cost} = z_0 + \underbrace{(- \quad - \quad -)}_{\text{some fn. in terms of } F\setminus\{v\} \text{ where coefficients are -ive}} + (\text{pos. coeff.})\, x_v + \underbrace{(\quad\quad)}_{\text{some fn. involving } x_{N\setminus F}}$

In every feasible soln. $x_{N\setminus F} = 0$

$x_v \leq 0$ x_v cannot be increased above 0.

$x_{F\setminus v} \geq 0$ Increasing these variables only reduces the cost.

z_0 is obtained by setting $x_{F\setminus v} = 0$, $x_v = 0$, $x_{N\setminus F} = 0$

$\therefore$ Optimum of aux. LP $\leq z_0$.

Now consider the tableau B' from where the variable x_v leave the basis.

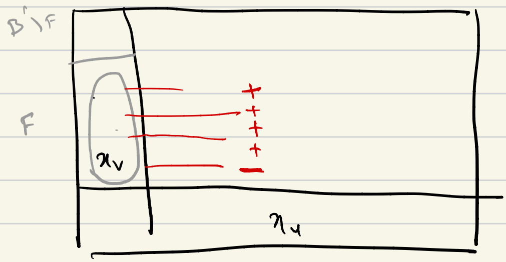
maximize $c^T x$

subject to: $Ax = b$

$x_{N \setminus F} = 0$

$x_v \leq 0$

$x_{F \setminus v} \geq 0$



B'

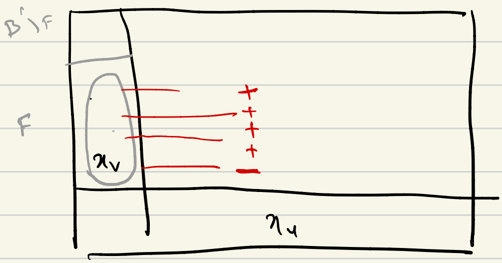
Claim: B' proves that the aux. LP is unbounded.

i) $x_u \in F \setminus \{x_v\}$. Increasing x_u will increase variables in F that are also in B', except for x_v.

This is because, we chose x_v to leave the basis. Hence Bland's rule will imply that all variables of F have a non-negative coefficient in the x_u column of B'.

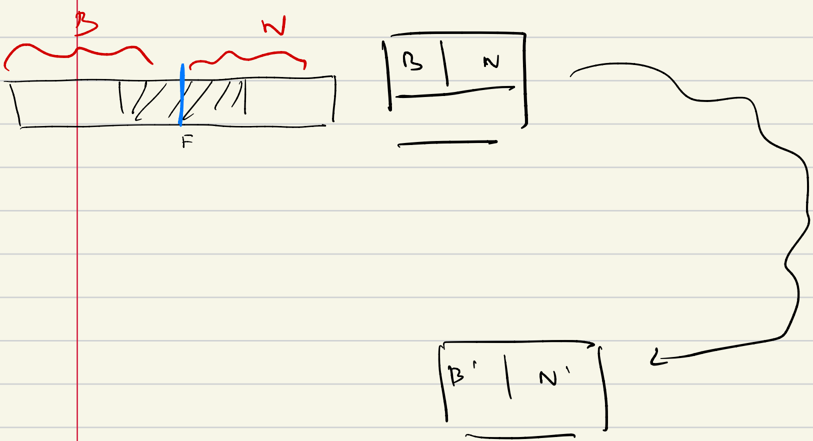
Notice that the aux. LP has no constraints on $x_{B'\setminus F}$.

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to:} \quad & Ax = b \\ & x_{N\setminus F} = 0 \\ & x_v \leq 0 \\ & x_{F\setminus v} \geq 0 \end{aligned}$$



This is because: $B \setminus F = B' \setminus F$,

$N \setminus F = N' \setminus F$



Moreover there is no constraint on $x_{B'} \setminus F$

Increasing x_u arbitrarily increases cost arbitrarily and still gives feasible solns to the aux. LP.

Contradiction:

i) Tableau B proves that aux. LP. is bounded.

ii) Tableau B' proves that aux. LP. is unbounded.

So our assumption that simplex using Bland's rule does not terminate is false.