# LINEAR OPTIMIZATION

LECTURE 10

22/05/2021

# Lecture 10: Simplex method - Part II

**Example:** Apply simplex on the following LP.

Maximize $x_1 + 2x_2$

Subject to:

$$
\begin{aligned}
x_1 + 3x_2 + x_3 &= 4 \\
2x_2 + x_3 &= 2 \\
x_1, \quad x_2, \quad x_3 &\geq 0
\end{aligned}
$$

How do we start?

- $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ $\rightarrow$ Find an initial feasible basis.

## Finding an initial feasible basis:

- This is a hard problem. Finding a basis is easy. But we need to ensure that it is feasible. A brute force enumeration of all bases is not useful, since it is expensive and as difficult as finding the optimum.

- We will make use of an auxiliary LP to detect feasibility and infer a feasible basis

## Back to example:

Maximize $x_1 + 2x_2$

Subject to:

$$x_1 + 3x_2 + x_3 = 4$$
$$2x_2 + x_3 = 2$$
$$x_1, x_2, x_3 \geq 0$$

- Add auxiliary variables $x_4, x_5$:

- Change constraints to:

$$x_4 = 4 - x_1 - 3x_2 - 3x_3$$
$$x_5 = 2 - 2x_2 - x_3$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

- Consider objective maximize $-x_4 - x_5$

Maximize $x_1 + 2x_2$
Subject to:

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_2 + x_3 &= 2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

LP

$\rightarrow$

Maximize $-x_4 - x_5$
subject to:

$$\begin{aligned} x_4 &= 4 - x_1 - 3x_2 - x_3 \\ x_5 &= 2 - 2x_2 - x_3 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Auxiliary LP

**Lemma:** LP is feasible iff optimum of auxiliary LP is 0.

**Proof:** Notice that the objective of auxiliary LP $\leq 0$.
- A feasible point of LP gives a point in aux. LP with cost 0.
- An optimum of aux. LP with cost 0 has $x_4 = x_5 = 0$. Hence the projection of optimum onto $\langle x_1, x_2, x_3 \rangle$ gives a feasible point of LP.

What is the advantage of auxiliary LP?

- A feasible basis can be detected easily. It is given by the extra variables.

For the above aux. LP

$$\begin{array}{ccccc} \left[\begin{matrix} 1 & 3 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \end{matrix}\right] \\ x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \end{array}$$

$B = \{4, 5\}$ is a basis. Moreover, substituting $x_1, x_2, x_3 = 0$ gives $x_4 = 4$, $x_5 = 2$. $\rightarrow$ a bfs.
$\langle 0, 0, 0, 4, 2 \rangle$

## What is the advantage of auxiliary LP ?

- A feasible basis can be detected easily. It is given by the extra variables.

For the above aux. LP
$$\begin{array}{ccccc} & & & & \\ \end{array}\begin{bmatrix} 1 & 3 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \end{bmatrix}$$
$$\quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$$

$B = \{4, 5\}$ is a basis. Moreover, substituting $x_1, x_2, x_3 = 0$ gives $x_4 = 4$, $x_5 = 2$. $\rightarrow$ a bfs.
$\langle 0, 0, 0, 4, 2 \rangle$

- Secondly, the cost of the aux LP $\leq 0$. Therefore the optimum is attained at a bfs. The simplex method on the aux LP. gives a bfs for the aux LP.

- Suppose $y^*$ is the bfs of aux LP that give optimum

Basic variables of $y^*$ do not include aux. variables
$\downarrow$
Same basis is feasible in LP

Some aux. variable is basic in the final tableau
$\downarrow$
(next page)

- Suppose $y^*$ is the bfs of aux. LP that gives optimum

Basic variables of $y^*$ do not include aux. variables

↓

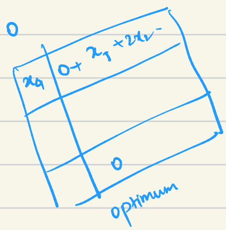
Same basis is feasible in LP

Some aux. variable is basic in the final tableau

↓

This is a degenerate case where the aux. variable which is basic also has value 0. The tableau can be rewritten so that the original variables are on the LHS and aux. variables are on the RHS.

- without changing the bfs.



Example:

Maximize $x_1 + 2x_2$

Subject to:

$$
\begin{aligned}
x_1 + 3x_2 + x_3 &= 4 \\
2x_2 + x_3 &= 2 \\
x_1,\ x_2,\ x_3 &\geq 0
\end{aligned}
$$

LP

$\rightarrow$

Maximize $-x_4 - x_5$

subject to:

$$
\begin{aligned}
x_4 &= 4 - x_1 - 3x_2 - x_3 \\
x_5 &= 2 \quad - 2x_2 - x_3 \\
x_1,\ x_2,\ x_3,\ x_4,\ x_5 &\geq 0
\end{aligned}
$$

Auxiliary LP

Solution to auxiliary LP:

$$
\begin{aligned}
x_4 &= 4 - x_1 - 3x_2 - x_3 \\
x_5 &= 2 \quad - 2x_2 - x_3 \\
\hline
z &= \underbrace{-6 + x_1 + 5x_2 + 2x_3}_{-x_4 - x_5}
\end{aligned}
\quad \xrightarrow[x_4 \downarrow]{x_1 \uparrow} \quad
\begin{aligned}
x_1 &= 4 - x_4 - 3x_2 - x_3 \\
x_5 &= 2 \quad - 2x_2 - x_3 \\
\hline
z &= -2 - x_4 + 2x_2 + x_3
\end{aligned}
$$

$$
\Big\downarrow \; x_5 \downarrow \quad x_3 \uparrow
$$

$$
\begin{aligned}
x_1 &= 2 - x_4 - x_2 - x_5 \\
x_3 &= 2 \quad - 2x_2 - x_5 \\
\hline
z &= -x_4 \quad - x_5 \\
\hline
\end{aligned}
$$

Optimum!

Optimum of aux. LP is 0 with basis $\{1, 3\}$

Notice that $\{1, 3\}$ is a feasible basis of original LP too.

Maximize $x_1 + 2x_2$

Subject to:

$$
\begin{aligned}
x_1 + 3x_2 + x_3 &= 4 \\
2x_2 + x_3 &= 2 \\
x_1,\ x_2,\ x_3 &\geq 0
\end{aligned}
$$

LP

We have inferred that $\{1, 3\}$ is a feasible basis.

Projecting the final tableau of the aux. LP to $\{x_1, x_2, x_3\}$ gives the initial tableau for LP:

$$
\begin{array}{rcl}
x_1 &=& 2 - x_2 \\
x_3 &=& 2 - 2x_2 \\
\hline
z &=& 2 + x_2
\end{array}
\xrightarrow[x_3 \downarrow]{x_2 \uparrow}
\begin{array}{rcl}
x_1 &=& 1 + x_3/2 \\
x_2 &=& 1 - x_3/2 \\
\hline
z &=& 3 - x_3/2
\end{array}
$$

Optimum

bfs : $\langle 1, 1, 0 \rangle$ gives optimum

Cost : 3

## Formalization of the simplex method:

At each step there is a bfs represented as a tableau.

$$
\begin{tikzcd}
B \arrow[r, leftrightarrow] \arrow[d, "\text{either stop with an answer or go to next step}"'] & T(B) \arrow[d] & \text{Tableau of } B \\
B' \arrow[r, leftrightarrow] & T(B')
\end{tikzcd}
$$

## Tableau of a bfs:

A simplex tableau determined by a feasible basis $B$ is a system of $m+1$ linear equations in variables

$$x_1, x_2, \dots, x_n, z$$

that has the same set of solutions as $Ax = b, x \geq 0$ and in matrix notation looks like:

$x_{k_1} = p_1 + \dots$
$x_{k_2} =$
$\vdots$
$x_{k_m}$

$$
\begin{aligned}
x_B &= p + Q x_N \\
\hline
z &= z_0 + r^T x_N
\end{aligned}
$$

$x_B \in \mathbb{R}^m$, $x_N \in \mathbb{R}^{n-m}$, $p \geq 0$ and $p \in \mathbb{R}^m$, $Q$ is an $m \times (n-m)$ matrix
$z_0 \in \mathbb{R}$, $r \in \mathbb{R}^{n-m}$

**Lemma 1:** For each feasible basis $B$, there exists exactly one tableau $T(B)$, and it is given by:

$$p = A_B^{-1} b$$

$$Q = -A_B^{-1} A_N$$

$$z_0 = c_B^T A_B^{-1} b$$

$$r^T = c_N^T - (c_B^T A_B^{-1} A_N)$$

**Proof:** We start with $Ax = b$, $x \geq 0$. We are given that $B$ is a feasible basis.

Rewriting $Ax = b$: $A_B x_B + A_N x_N = b$

$A_B$: $m \times m$ $A_N$: $m \times (n-m)$

$$A_B x_B = b - A_N x_N$$

We know $A_B$ has rank $m$. Hence there is an inverse $A_B^{-1}$.

$$A_B^{-1} A_B x_B = A_B^{-1} b - A_B^{-1} A_N x_N$$

$$x_B = A_B^{-1} b - A_B^{-1} A_N x_N$$

This gives a tableau with:

$$p = A_B^{-1} b \qquad Q = -A_B^{-1} A_N$$

$$x_B = A_B^{-1} b - A_B^{-1} A_N x_N$$

$c_B$: the coordinate of $c$ corresponding to basic variables $B$

$c_N$: non-basic

What about cost $c^T$?

$$c^T x = c_B^T x_B + c_N^T x_N$$

$$= c_B^T (A_B^{-1} b - A_B^{-1} A_N x_N) + c_N^T x_N$$

$$= \underbrace{c_B^T A_B^{-1} b}_{z_0} + \underbrace{(c_N^T - c_B^T A_B^{-1} A_N)}_{r^T} x_N$$

This shows that for every $B$ there is a tableau $T(B)$

<u>Uniqueness:</u>

$$x_B = p' + Q' x_N$$

$$z = z_0' + r'^T x_N$$

It is enough to show $p = p'$ and $Q = Q'$.

$$x_B = p + Q x_N \quad \}^m$$

$$x_B = p' + Q' x_N \quad \}^m$$

Subtracting coordinate wise:

$$(p - p') + (Q - Q') x_N = 0 \quad \text{for all values of } x_N.$$

Now putting $x_N = 0$ gives $p = p'$

To show $Q = Q'$

$$\left( \begin{bmatrix} & & \end{bmatrix} - \begin{bmatrix} & & \end{bmatrix}' \right) \underbrace{\begin{bmatrix} x_N^1 \\ x_N^2 \\ \vdots \\ x_N^{n-m} \end{bmatrix}}_{x_N} = 0$$

$$x_N^1 \left( Q^1 - Q'^1 \right) + x_N^2 \left( Q^2 - Q'^2 \right) + \cdots + x_N^{n-m} \left( Q^{n-m} - Q'^{n-m} \right) = 0$$

Putting $x_N^1 = 1$ and others $0$ gives $Q^1 = Q'^1$

$\vdots$

$x_N^i = 1$ others $0$ — $Q^i = Q'^i$

This shows $Q = Q'$.

**Lemma 2:** If $T(B)$ is a simplex tableau s.t. $r \leq 0$, then the optimum is $z_0$, attained at the bfs corresponding to $B$.

$$Z = z_0 + r_1 x_N^1 + r_2 x_N^2 \cdots r_N^{n-m} x_N^{n-m}.$$

We stop when every $r_i \leq 0$

**Proof:** At any feasible soln. $y$, the cost is given by:

$$z_0 + r^T y_N$$

We know $y_N \geq 0$

$\therefore$ since $r \leq 0$, $z_0 + r^T y_N \leq z_0$

Now, the bfs corresponding to the tableau has cost $z_0$. Hence it is an optimum.

## Pivoting step:

Suppose we have a tableau $T(B)$ s.t. $r$ is not $\leq 0$ at every coordinate.

How do we choose an $x_v$ that enters the basis and an $x_u$ that leaves the basis?

Assume that $B = \{k_1, k_2, \ldots, k_m\}$ $N = \{l_1, l_2, \ldots, l_{n-m}\}$

$$
\begin{aligned}
k_1 &= p_1 + q_{11} l_1 + q_{12} l_2 + \cdots + q_{1,n-m} l_{n-m} \\
&\ \vdots \\
k_i &= p_i + q_{i1} l_1 + q_{i2} l_2 + \cdots + q_{i,n-m} l_{n-m} \\
&\ \vdots \\
k_m &= p_m + q_{m1} l_1 + \cdots + q_{m,n-m} l_{n-m} \\
\hline
z &= z_0 + r_1 l_1 + r_2 l_2 + \cdots + r_{n-m} l_{n-m}
\end{aligned}
$$

Suppose we want to check if $l_i$ can enter the basis:

- $r_i > 0$
- Consider column corresponding to $l_i$ $\begin{bmatrix} q_{1i} \\ q_{2i} \\ \vdots \\ q_{mi} \end{bmatrix}$
- If all coordinate $q_{ji} \geq 0$, then LP is unbounded.
- Otherwise we will pick a variable $k_j$ which gives the tightest constraint for $l_i$.

$$
\begin{aligned}
k_1 &= p_1 + q_{11} l_1 + q_{12} l_2 + \cdots + q_{1,n-m} l_{n-m} \\
k_i &= p_i + q_{i1} l_1 + q_{i2} l_2 + \cdots + q_{i,n-m} l_{n-m} \\
&\ \ \vdots \\
k_m &= p_m + q_{m1} l_1 + \cdots + q_{m,n-m} l_{n-m} \\
\hline
z &= z_0 + r_1 l_1 + r_2 l_2 + \cdots + r_{n-m} l_{n-m}
\end{aligned}
$$

$l_i$

$$
\begin{array}{cc}
 & \begin{bmatrix} q_{1i} \\ q_{2i} \\ \vdots \\ q_{mi} \end{bmatrix} \\
x_{k_2} & \qquad q_{2i} < 0
\end{array}
$$

Look at all rows $j$ s.t. $q_{ji} < 0$

$$x_{k_j} = p_j + q_{ji} l_i + \{ \quad \}$$

This gives a bound $\dfrac{-p_j}{q_{ji}}$ for $l_i$

Choose an $\alpha$ which gives: $\min \left\{ \dfrac{-p_j}{q_{ji}} : q_{ji} < 0 \right\}$

$k_\alpha$ leaves the basis, $l_i$ enters basis.

$x_u = k_\alpha$ $\qquad$ $x_v = l_i$

**Lemma:** $B' = (B \setminus \{u\}) \cup \{v\}$ is a feasible basis.

**Proof:**

$$x_B = p + Q x_N$$

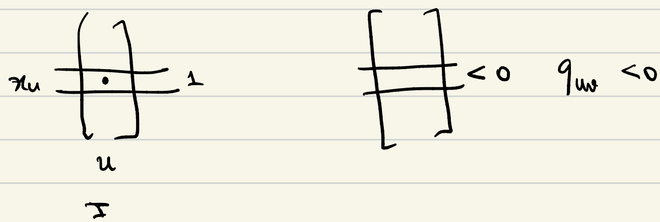
$$I^m x_B - Q x_N = p$$

$m \times m$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & \cdots & \\ & & & \cdots & 1 \end{bmatrix}}_{B} x_B - \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1,n-m} \\ \vdots & & & \\ q_{m1} & q_{m2} & \cdots & q_{m,n-m} \end{bmatrix} x_N = \begin{bmatrix} p_1 \\ \vdots \\ p_m \end{bmatrix}$$

$u$ $\qquad$ $v$

i) The set of columns obtained by removing the column corresponding to $u$ in $I^{m \times m}$ and adding the $-Q^v$ from $Q$, gives a set of $m$ columns that are linearly independent.




Combination of other columns in $I$ cannot give $Q^v$

ii) $B \setminus \{u\} + \{v\}$ gives a feasible soln. by putting $N \cup \{u\} = 0$. This is because of our choice made in the last slide (detailed proof left as exercise)

(iii) $B \setminus \{u\} \cup \{v\}$ is linearly independent in the original system as well.

$$x_B - \theta x_N = p$$

Start with A: $\begin{bmatrix} & A & \\ & & \end{bmatrix}$ and perform Gaussian elimination.

$$A_B x_B + A_N x_N$$

- Multiplying by $A_B^{-1}$: $\overset{A_B^{-1}}{\begin{bmatrix} & \\ & \end{bmatrix}} \begin{bmatrix} A_B & \big| & A_N \end{bmatrix}$

is equivalent to replacing rows of A by a weighted combination of its rows given by $A_B^{-1}$.

- Now consider a set of columns that are linearly independent after applying the Gaussian elimination. Then they have to be linearly independent in the original matrix as well.
  - ↳ Restrict matrix to only this set. Say there are $k$ columns. If they are not lin. independent, then rank of original matrix $< k$.

- There are $< k$ lin. independent rows in original system.

- G.E. preserves the linearly independent rows.
  Hence rank is the same after G.E.

- So the columns $B \setminus \{u\} \cup \{v\}$ will have full rank in the original system since they have full rank in the eliminated system.

**Lemma:** If there exist a non-basic variable $x_v$ s.t. $r_v > 0$ and the column $Q^v$ has only non-negative entries, the LP is unbounded.

**Proof:** Putting $x_v = t$ and other non-basic variables to 0 gives a feasible soln. for all $t \geq 0$.

$$x_{k_i} = p \cdot \left[ + \cdot \overset{t}{x_v} \ \overbrace{\;-\;}^{0} \right]$$

Moreover since $r^v > 0$. Increasing values of $x_v$ will increase cost.

Hence there is no bound.

**Summary:**

- Finding an initial basis using aux. LP.
- Formalization of simplex
    - We have assumed that simplex terminates.

What is the proof of this?

↳ Next lecture: some pivoting rules that ensure termination.