LINEAR OPTIMIZATION

LOGISTICS

- 4 credit course
- No special prerequisites
- Evaluation:

  2 Quizzes + Midsem + Endsem

  ↑

  may take a viva

- Book:

  Understanding and Using Linear Programming

  Jiři Matoušek, Bernd Gärtner

Today's Lecture

- What is a linear program?
- Examples

REFERENCE: Chapters 1 & 2 of:

Understanding and Using Linear Programming
Jiři Matoušek, Bernd Gärtner

EXAMPLE 1: Consider the following problem.

The operator of a restaurant wants to make a healthy dish out of carrots, cabbages and pickled cucumber. Here is some nutritional info and the cost of each ingredient.

Food	Carrot	Cabbage	Cucumber	Required per dish
Vitamin A [mg/kg]	35	0.5	0.5	0.5 mg
Vitamin C [mg/kg]	60	300	10	15 mg
Fibre [g/kg]	30	20	10	4 g
price [€/kg]	0.75	0.5	0.15	

At what minimum price per dish can the nutrition requirements be satisfied?

Food	Carrot	Cabbage	Cucumber	Required per dish
Vitamin A [mg / kg]	35	0.5	0.5	0.5 mg
Vitamin C [mg / kg]	60	300	10	15 mg
Fibre [g / kg]	30	20	10	4 g
price [€ / kg]	0.75	0.5	0.15	

Step 1: Set up the variables.

x_1 : weight of carrot in kg per dish

x_2 : weight of cabbage in kg per dish

x_3 : weight of cucumber in kg per dish.

$x_1, x_2, x_3 \in \mathbb{R}$ (reals)

Food	Carrot	Cabbage	Cucumber	Required per dish
Vitamin A [mg / kg]	35	0.5	0.5	0.5 mg
Vitamin C [mg / kg]	60	300	10	15 mg
Fibre [g / kg]	30	20	10	4 g
price [€ / kg]	0.75	0.5	0.15	

Step 2: Set up the constraints.

$$35x_1 + 0.5x_2 + 0.5x_3 \geq 0.5$$

$$60x_1 + 300x_2 + 10x_3 \geq 15$$

$$30x_1 + 20x_2 + 10x_3 \geq 4$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0$$

Food	Carrot	Cabbage	Cucumber	Required per dish
Vitamin A [mg / kg]	35	0.5	0.5	0.5 mg
Vitamin C [mg / kg]	60	300	10	15 mg
Fibre [g / kg]	30	20	10	4 g
price [€ / kg]	0.75	0.5	0.15	

Step 3: Minimize the cost per dish

$$\text{Minimize} \quad 0.75\, x_1 + 0.5\, x_2 + 0.15\, x_3$$

Food	Carrot	Cabbage	Cucumber	Required per dish
Vitamin A [mg / kg]	35	0.5	0.5	0.5 mg
Vitamin C [mg / kg]	60	300	10	15 mg
Fibre [g / kg]	30	20	10	4 g
price [€ / kg]	0.75	0.5	0.15	

Linear Program (LP):

Minimize $\quad 0.75\, x_1 + 0.5\, x_2 + 0.15\, x_3$

subject to :

$$35\, x_1 + 0.5\, x_2 + 0.5\, x_3 \geqslant 0.5$$

$$60\, x_1 + 300\, x_2 + 10\, x_3 \geqslant 15$$

$$30\, x_1 + 20\, x_2 + 10\, x_3 \geqslant 4$$

$$x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0$$

Linear Program (LP):

Minimize $\quad 0.75\,x_1 + 0.5\,x_2 + 0.15\,x_3$

subject to :

$$35\,x_1 + 0.5\,x_2 + 0.5\,x_3 \geqslant 0.5$$

$$60\,x_1 + 300\,x_2 + 10\,x_3 \geqslant 15$$

$$30\,x_1 + 20\,x_2 + 10\,x_3 \geqslant 4$$

$$x_1 \geqslant 0,\quad x_2 \geqslant 0,\quad x_3 \geqslant 0$$

Solution:

$x_1 = 0.0095$ $\qquad x_2 = 0.038$ $\qquad x_3 = 0.29$

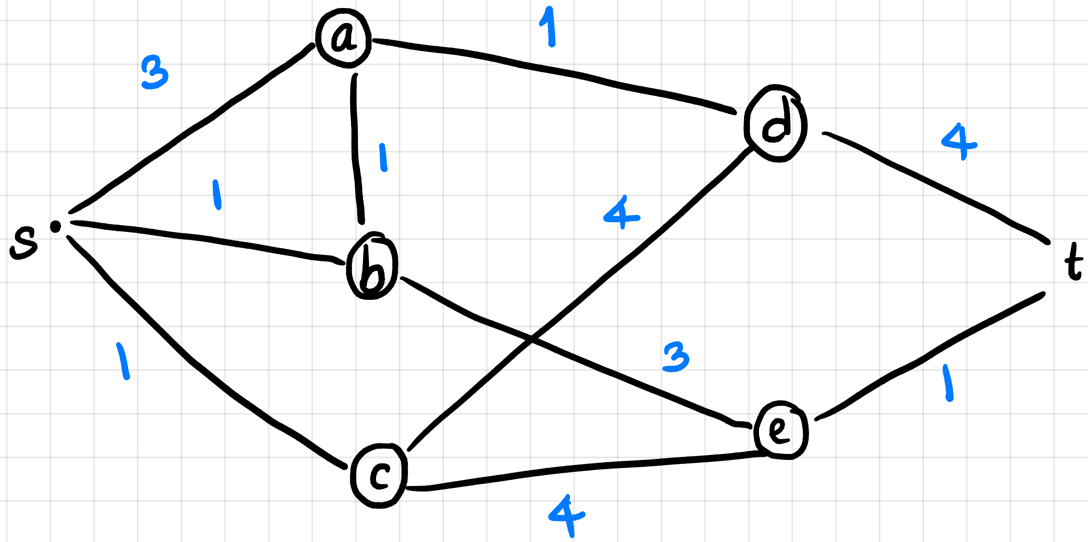
Carrot 9.5 g $\qquad$ Cabbage 38 g $\qquad$ Cucumber 290 g

Price per dish: € 0.07

IN THIS COURSE ...

- 1. Algorithms to solve LPs.
  - Simplex, Primal-dual algorithm
- 2. Some theoretical properties of LPs.
  - Convex polyhedra, duality
- 3. Applications of LP
  - Zero-sum games
  - Matchings and vertex covers in graphs

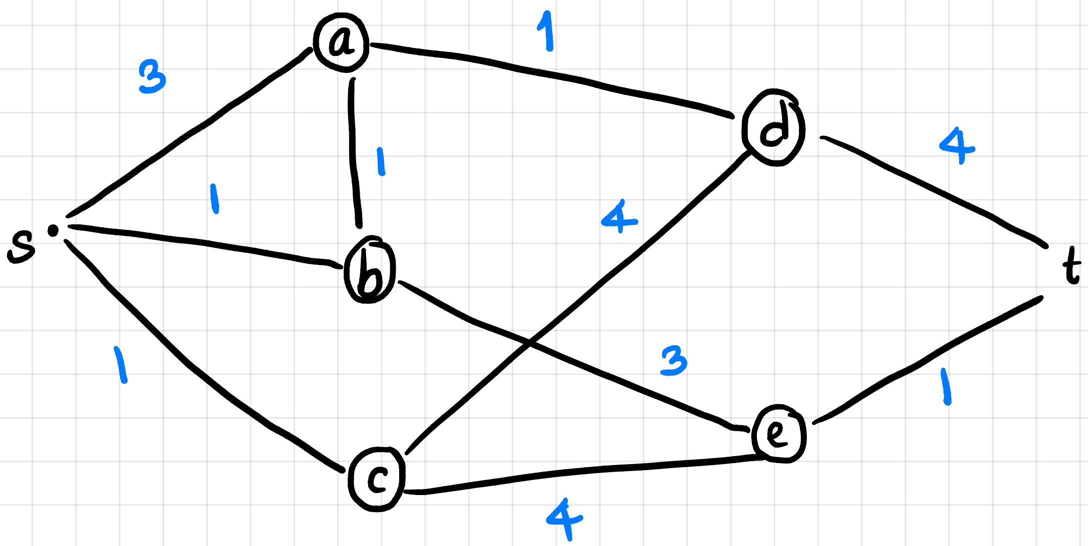
EXAMPLE 2: Network flows



- The goal is to transfer files from s to t using the network on the right.
- On each edge (link), data can flow in both directions, but not simultaneously.
- Numbers next to the links show the maximum transfer rate in that link in Mbit/s
- Nodes do not store any data.

Question: What is the maximum transfer rate?

What is the problem?

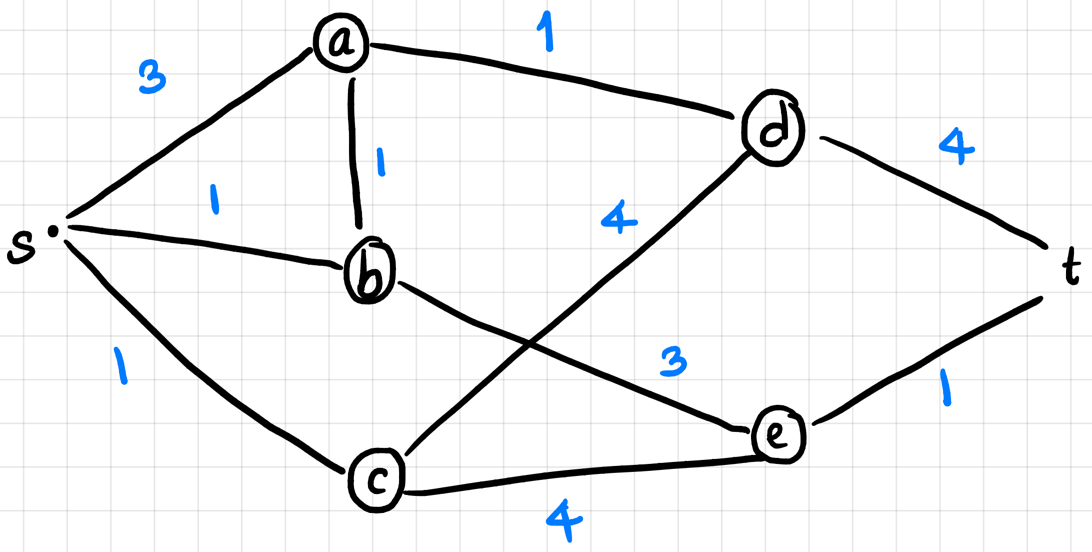


- Full capacity of a link may not be possible to utilize.
(eg. @)

- Secondly, we need to choose the direction of data flow for each link.

  - Transfer amount + direction for each link.

An LP for Network flow:



if $x_{ce} = 3$

$x_{ce} = -2$

Step 1: Choose the variables

x_{sa}, x_{sb}, x_{sc}

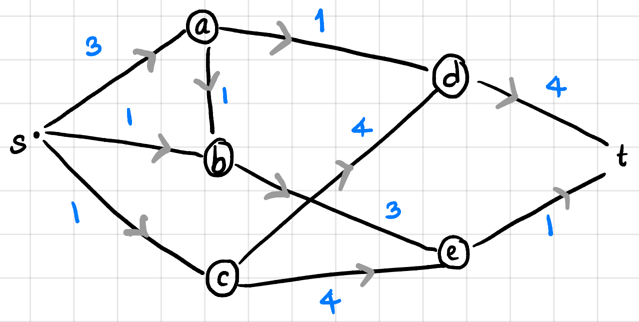
x_{ad}, x_{ab}

x_{be}

x_{cd}, x_{ce}

x_{dt}

x_{et}



Step 1: Choose the variables

x_{sa}, x_{sb}, x_{sc} x_{ad}, x_{ab}

x_{be} x_{cd}, x_{ce}

x_{dt} x_{et}

Step 3: Formulate the objective

Maximize $x_{sa} + x_{sb} + x_{sc}$

Step 2: Write the constraints

$-3 \leq x_{sa} \leq 3$ $-1 \leq x_{ad} \leq 1$

$-1 \leq x_{sb} \leq 1$ $-3 \leq x_{be} \leq 3$

$-1 \leq x_{sc} \leq 1$ $-4 \leq x_{cd} \leq 4$

$-1 \leq x_{ab} \leq 1$ $-4 \leq x_{ce} \leq 4$

$-4 \leq x_{dt} \leq 4$ $-1 \leq x_{et} \leq 1$

$$x_{sa} = x_{ab} + x_{ad}$$

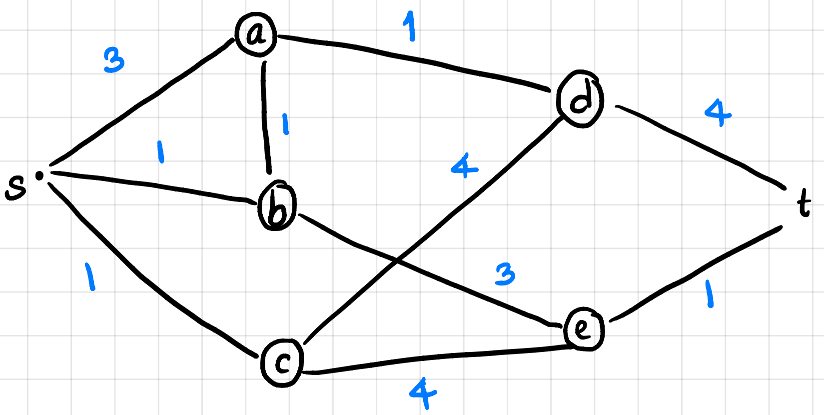
$$x_{sb} + x_{ab} = x_{be}$$

$$x_{sc} = x_{cd} + x_{ce}$$

$$x_{ad} + x_{cd} = x_{dt}$$

$$x_{be} + x_{ce} = x_{et}$$

Final LP :



Maximize $x_{sa} + x_{sb} + x_{sc}$

subject to:

$$-3 \le x_{sa} \le 3 \qquad -1 \le x_{ad} \le 1$$

$$-1 \le x_{sb} \le 1 \qquad -3 \le x_{be} \le 3$$

$$-1 \le x_{sc} \le 1 \qquad -4 \le x_{cd} \le 4$$

$$-1 \le x_{ab} \le 1 \qquad -4 \le x_{ce} \le 4$$

$$-4 \le x_{dt} \le 4 \qquad -1 \le x_{et} \le 1$$

$$x_{sa} = x_{ab} + x_{ad}$$

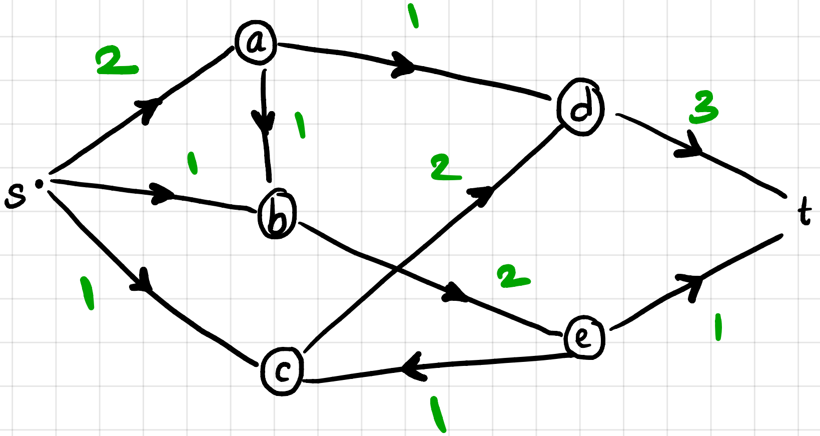
$$x_{sb} + x_{ab} = x_{be}$$

$$x_{sc} = x_{cd} + x_{ce}$$

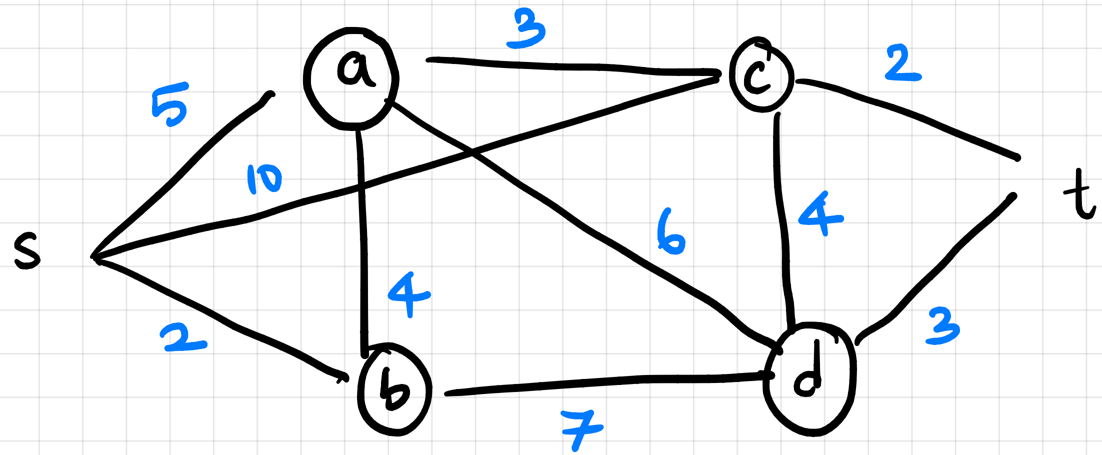
$$x_{ad} + x_{cd} = x_{dt}$$

$$x_{be} + x_{ce} = x_{et}$$

Optimal solution:



Problem: Write an LP for the following network flow problem using the variables given below.

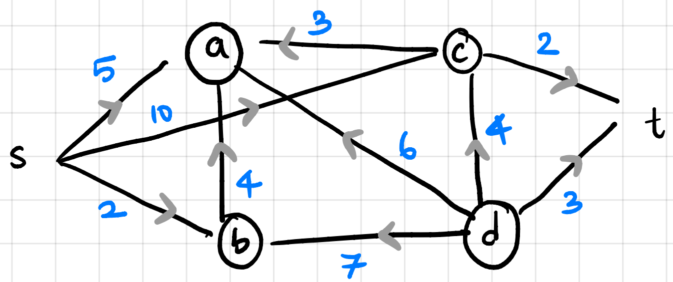


x_{sa}, x_{sc}, x_{sb}

x_{ba}

x_{ca}, x_{ct}

x_{da}, x_{db}, x_{dc}, x_{dt}



x_{sa}, x_{sc}, x_{sb}

x_{ba}

x_{ca}, x_{ct}

x_{da}, x_{db}, x_{dc}, x_{dt}

Maximize $\quad x_{sa} + x_{sb} + x_{sc}$

Subject to:

$-5 \leq x_{sa} \leq 5$	$-2 \leq x_{ct} \leq 2$
$-10 \leq x_{sc} \leq 10$	$-6 \leq x_{da} \leq 6$
$-2 \leq x_{sb} \leq 2$	$-7 \leq x_{db} \leq 7$
$-3 \leq x_{ca} \leq 3$	$-4 \leq x_{dc} \leq 4$
$-4 \leq x_{ba} \leq 4$	$-3 \leq x_{dt} \leq 3$

$$x_{sa} + x_{ba} + x_{ca} + x_{da} = 0$$

$$x_{sb} + x_{db} = x_{ba}$$

$$x_{sc} + x_{dc} = x_{ca} + x_{ct}$$

$$0 = x_{db} + x_{da} + x_{dc} + x_{dt}$$

EXAMPLE 3: Production schedule optimization.

- Demand of a product in a year: $d_1, d_2, d_3, \dots, d_{12}$

d_i : demand for month i (in tons)

- Storage cost (of surplus) : €20 per month for 1 ton
- $\left.\begin{array}{l}\text{Changing the production by} \\ \text{1 ton from month } i-1 \text{ to } i\end{array}\right\}$: €50

Let x_i be the production in tons for month i.

Find a production schedule with the minimum cost that meets the demands. Write an LP for this problem.

Cost : Cost of storing surplus + cost of changing production

To calculate cost due to surplus, introduce new variables $s_1, s_2, \dots s_{12}$

$$s_i = x_i + s_{i-1} - d_i$$

Surplus at month i

Production at i

Surplus carried over from i-1

demand in month i

Assume $s_0 = 0$

Constraints:

$$x_i + s_{i-1} \geq d_i$$

Meet the demand

$$S_i = x_i + S_{i-1} - d_i$$

$$S_0 = 0$$

Cost:

$$\sum_{i=1}^{12} S_i \cdot 20 + \sum_{i=1}^{12} |x_i - x_{i-1}| \cdot 50$$

with $x_0 = 0$

Cost due to surplus

cost due to production change

$$|\underline{x_i - x_{i-1}}|$$

Constraints:

$$x_i + s_{i-1} \geq d_i$$

$$s_i = x_i + s_{i-1} - d_i$$

$$s_0 = 0$$

Cost:

$$\sum_{i=1}^{12} s_i \cdot 20 + \sum_{i=1}^{12} |x_i - x_{i-1}| \cdot 50$$

with $x_0 = 0$

Cost due to surplus

cost due to production change

Problem: This is not an LP as Cost contains modulus $|x_i - x_{i-1}|$

Dealing with modulus:

Introduce new variables y_i, z_i

Add constraints:

$$\boxed{\begin{array}{ll} y_i \geq 0 & z_i \geq 0 \\ x_i - x_{i-1} = y_i - z_i & \end{array}}$$

$x_i - x_{i-1} = 5$ $\quad y_i = 5$, $z_i : 0$

$x_i - x_{i-1} = -5$ $\quad y_i = 0$, $z_i = 5$

Idea: When $x_i - x_{i-1} \geq 0$, we should have $y_i = x_i - x_{i-1}$

$z_i = 0$

When $x_i - x_{i-1} < 0$, we should have $y_i = 0$

$-z_i = x_i - x_{i-1}$

Add constraints:

$$\boxed{\begin{array}{ll} y_i \geq 0 & z_i \geq 0 \\ x_i - x_{i-1} = y_i - z_i & \end{array}}$$

Idea: When $x_i - x_{i-1} \geq 0$, we should have $y_i = x_i - x_{i-1}$, $z_i = 0$

When $x_i - x_{i-1} < 0$, we should have $y_i = 0$, $-z_i = x_i - x_{i-1}$

Modify the Objective accordingly

Previous cost: $$\sum_{i=1}^{12} S_i \cdot 20 + \sum_{i=1}^{12} |x_i - x_{i-1}| \cdot 50$$

New cost: $$\sum_{i=1}^{12} S_i \cdot 20 + \sum_{i=1}^{12} y_i \cdot 50 + \sum_{i=1}^{12} z_i \cdot 50$$

Final LP:

Minimize $\sum_{i=1}^{12} s_i \cdot 20 + \sum_{i=1}^{12} y_i \cdot 50 + \sum_{i=1}^{12} z_i \cdot 50$

Subject to.

$$x_i + s_{i-1} \geq d_i$$

$$s_i = x_i + s_{i-1} - d_i$$

$$s_0 = 0$$

$$s_i \geq 0$$

$$x_i - x_{i-1} = y_i - z_i$$

$$y_i \geq 0 \qquad z_i \geq 0$$

$x_1 = 5 \quad x_2 = 10$

$y_2 - z_2 = 5$

y_2	z_2
100	95
10	5
⋮	
5	0

$x_1 = 10 \quad x_2 = 5$

$y_2 - z_2 = -5$

$y_2 = 0$

$z_2 = 5$

Minimize $\sum_{i=1}^{12} s_i \cdot 20 + \sum_{i=1}^{12} y_i \cdot 50 + \sum_{i=1}^{12} z_i \cdot 50$

Subject to.

$$x_i + s_{i-1} \geq d_i$$

$$s_i = x_i + s_{i-1} - d_i$$

$$s_0 = 0$$

$$s_i \geq 0$$

$$x_i - x_{i-1} = y_i - z_i$$

$$y_i \geq 0 \qquad z_i \geq 0$$

Claim: In the optimal solution,

- either $y_i = 0$ or $z_i = 0$
- if $x_i - x_{i-1} \geq 0$, $y_i = x_i - x_{i-1}$, $z_i = 0$

  else $-z_i = x_i - x_{i-1}$, $y_i = 0$

Proof: Suppose σ^* is an optimal soln.

Define τ^*:

$$\tau^*(x_i) = \sigma^*(x_i)$$

$$\tau^*(s_i) = \sigma^*(s_i)$$

if $x_i - x_{i-1} \geq 0$, $\tau^*(y_i) = x_i - x_{i-1}$, $\tau^*(z_i) = 0$

else $-\tau^*(z_i) = x_i - x_{i-1}$, $\tau^*(y_i) = 0$

τ^* has smaller cost.

EXAMPLE 4: Fitting a line

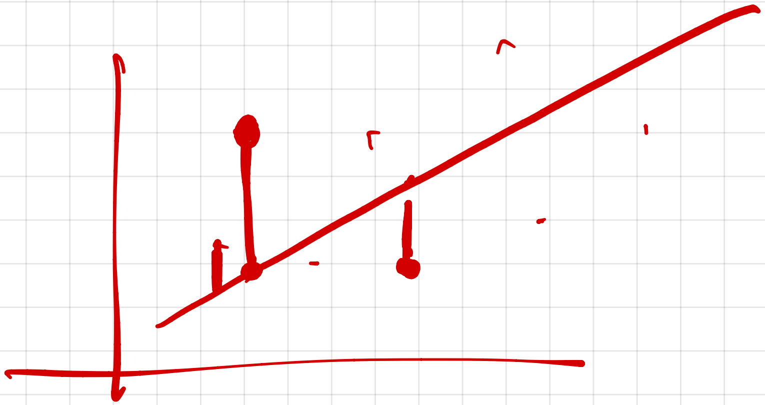
Given a set of 'n' points:

$$(x_1, y_1), (x_2, y_2) \ldots, (x_n, y_n)$$

Find a line $y = ax + b$ that minimizes the total error:

$$\sum_{i=1}^{n} |ax_i + b - y_i|$$

Write an LP for this problem.



LP:

Minimize $\sum_{i=1}^{n} e_i$

$e_i = |ax_i + b - y_i|$

-5

Subject to:

$e_i \geq 0$

$e_i \geq -5$

$e_i \geq 5$

$e_i \geq ax_i + b - y_i$

$i = 1, 2, \dots, n$

$e_i \geq -(ax_i + b - y_i)$

Claim: In the optimal solution, $e_i = |ax_i + b - y_i|$

minimize: $|y_i - (ax_i - b)|$

p_i q_i

$p_i \geq 0, q_i \geq 0$

$y_i - (ax_i + b) = p_i - q_i$

LINEAR PROGRAM : General form

Maximize $\quad c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

Subject to

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1$$

$$\vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m$$

n variables

m constraints

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

A $\qquad b$

maximize $\quad c^T x$

subject to $\quad Ax \leq b$

$$x \in \mathbb{R}^n$$

$c : n \times 1$

$A : m \times n$

$x : n \times 1$

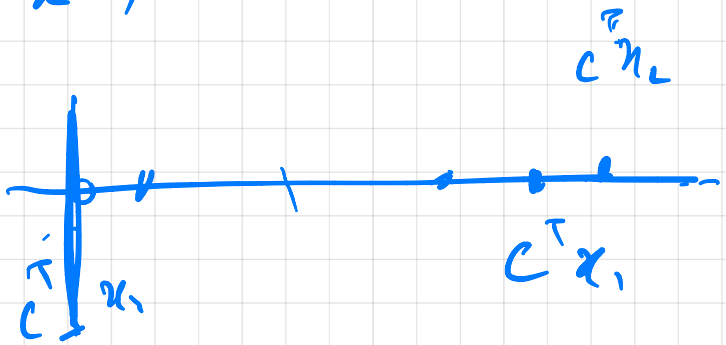
$b : m \times 1$

c, A, b are matrices over reals.

Question: How to convert a minimization problem to a maximization problem?

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \le b \end{aligned}$$

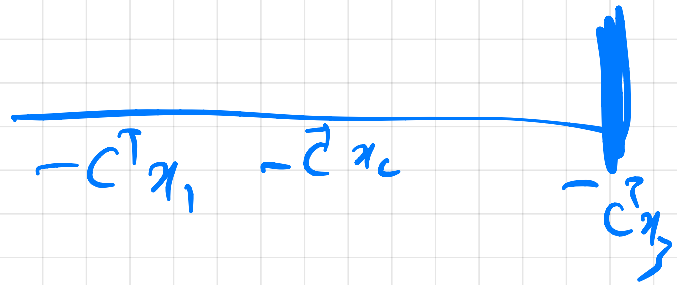
$x^*, \; -K$



$\downarrow$

$$\begin{aligned} &\text{maximize} && -c^T x \\ &\text{subj. to} && Ax \le b \end{aligned}$$

$x^*, \; K$



Question: How to convert a minimization problem to a maximization problem?

$$\text{minimize} \quad c^T x$$
$$\text{subject to} \quad Ax \leq b$$

Solution: Consider:

$$\text{maximize} \quad -c^T x$$
$$\text{subject to} \quad Ax \leq b$$

If K is optimum cost and σ^* is the optimum for the converted problem, then $-K$ is the optimum cost and σ^* is the optimum for original problem.

SUMMARY:

- What is LP.
- Examples: optimized diet

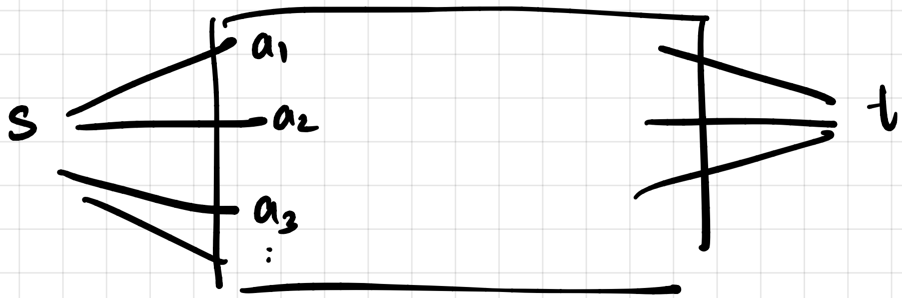
  network flows

  production schedule

  fitting a line

  dealing with modulus

Question: Consider a network flow problem.



- Assume that the variables involving 's' are $x_{sa_1}, x_{sa_2} \dots x_{sa_k}$
- Recall the LP we wrote for this problem.
- Can any of $x_{sa_1} \dots x_{sa_k}$ have a negative value in the optimum solution?