

APPLICATIONS OF LP: ZERO-SUM GAMES - Part 3:

Recall:

- For pure strategies, $\max \min \leq \min \max$, and equality is attained iff there are saddle points (Part 1)
- Mixed strategies, and an LP to compute $\max \min$ over mixed strategies. (Part 2)

Today: We will prove that over mixed strategies: $\max \min = \min \max$.

LP for max min:

maximize x_0

subject to: $x_0 \leq 0 \cdot x_1 + 1 \cdot x_2 - 1 \cdot x_3$

$$x_0 \leq -1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3$$

$$x_0 \leq +1 \cdot x_1 - 1 \cdot x_2 + 0 \cdot x_3$$

$$x_1 + x_2 + x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

x_1 Rock

x_2 Paper

x_3 Scissors

	y_1 Rock	y_2 Paper	y_3 Scissors
x_1 Rock	0	-1	+1
x_2 Paper	+1	0	-1
x_3 Scissors	-1	+1	0

LP for min max:

minimize y_0

subject to: $0 \cdot y_1 - 1 \cdot y_2 + 1 \cdot y_3 \leq y_0$

$$1 \cdot y_1 + 0 \cdot y_2 - 1 \cdot y_3 \leq y_0$$

$$-1 \cdot y_1 + 1 \cdot y_2 + 0 \cdot y_3 \leq y_0$$

$$y_1 + y_2 + y_3 = 1$$

$$y_1, y_2, y_3 \geq 0$$

Dual of the LP for max min:

maximize x_0

subject to:

$$y_1 \times x_0 \leq 0 \cdot x_1 + 1 \cdot x_2 - 1 \cdot x_3$$
$$y_2 \times x_0 \leq -1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3$$
$$y_3 \times x_0 \leq +1 \cdot x_1 - 1 \cdot x_2 + 0 \cdot x_3$$

	Rock	Paper	Scissors
Rock	0	-1	+1
Paper	+1	0	-1
Scissors	-1	+1	0

$$y_0 \times x_1 + x_2 + x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

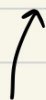


minimize y_0

subject to: $y_1 + y_2 + y_3 = 1$ (x_0)

$$0 \cdot y_1 + 1 \cdot y_2 - 1 \cdot y_3 + y_0 \geq 0 \quad (x_1)$$
$$-1 \cdot y_1 + 0 \cdot y_2 + 1 \cdot y_3 + y_0 \geq 0 \quad (x_2)$$
$$+1 \cdot y_1 - 1 \cdot y_2 + 0 \cdot y_3 + y_0 \geq 0 \quad (x_3)$$

$$y_1, y_2, y_3 \geq 0 \quad (\text{inequalities in primal})$$



This is exactly the LP for min-max!

- Since primal is bounded and feasible, there is an optimum.

Hence primal-optimum = dual-optimum, i.e. max min = min max!

In general: Given game M

max - min LP:

maximize x_0

subject to:

$$x_0 \leq M_{11}x_1 + M_{21}x_2 + \dots + M_{m1}x_m$$

$$x_0 \leq M_{12}x_1 + M_{22}x_2 + \dots + M_{m2}x_m$$

\vdots

$$x_0 \leq M_{1n}x_1 + M_{2n}x_2 + \dots + M_{mn}x_m$$

$$x_1 + x_2 + \dots + x_m = 1$$

$$x_1, x_2, \dots, x_m \geq 0$$

min - max LP:

minimize y_0

subject to:

$$y_0 \geq M_{11}y_1 + M_{12}y_2 + \dots + M_{1n}y_n$$

$$y_0 \geq M_{21}y_1 + M_{22}y_2 + \dots + M_{2n}y_n$$

\vdots

$$y_0 \geq M_{m1}y_1 + M_{m2}y_2 + \dots + M_{mn}y_n$$

$$y_1 + y_2 + \dots + y_n = 1$$

$$y_1, y_2, \dots, y_n \geq 0$$

These two LPs are duals of each other.

Minimax theorem: max - min = min - max over mixed strategies

There exist mixed strategies \bar{x} and \bar{y} for Max and Min s.t.:

- \bar{y} is the best response to \bar{x} and
- \bar{x} is the best response to \bar{y}

There is a Nash equilibrium over mixed strategies for zero-sum games.