APPLICATIONS OF LP: ZERO-SUM GAMES - Part 3:

Recall:

- For pure strategies, max min $\leq$ min max, and equality is attained iff there are saddle points (Part 1)

- Mixed strategies, and an LP to compute max min over mixed strategies. (Part 2)

Today: We will prove that over mixed strategies: max min = min max.

LP for max min:

	y_1 Rock	y_2 Paper	y_3 Scissors
x_1 Rock	0	−1	+1
x_2 Paper	+1	0	−1
x_3 Scissors	−1	+1	0

maximize $\quad x_0$

subject to:

$$\begin{aligned}
x_0 &\leq 0\cdot x_1 + 1\cdot x_2 - 1\cdot x_3 \\
x_0 &\leq -1\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 \\
x_0 &\leq +1\cdot x_1 - 1\cdot x_2 + 0\cdot x_3 \\
x_1 + x_2 + x_3 &= 1 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

LP for min max:

minimize $\quad y_0$

subject to:

$$\begin{aligned}
0\cdot y_1 - 1\cdot y_2 + 1\cdot y_3 &\leq y_0 \\
1\cdot y_1 + 0\, y_2 - 1\cdot y_3 &\leq y_0 \\
-1\, y_1 + 1\, y_2 + 0\, y_3 &\leq y_0 \\
y_1 + y_2 + y_3 &= 1 \\
y_1, y_2, y_3 &\geq 0
\end{aligned}$$

Dual of the LP for max min:

	Rock	Paper	Scissors
Rock	0	−1	+1
Paper	+1	0	−1
Scissors	−1	+1	0

maximize x_0

subject to:

$$
\begin{aligned}
y_1 \times \quad x_0 &\leq 0 \cdot x_1 + 1 \cdot x_2 - 1 \cdot x_3 \\
y_2 \times \quad x_0 &\leq -1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 \\
y_3 \times \quad x_0 &\leq +1 \cdot x_1 - 1 \cdot x_2 + 0 \cdot x_3
\end{aligned}
$$

$$y_0 \times \quad x_1 + x_2 + x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

minimize y_0

subject to:

$$y_1 + y_2 + y_3 = 1 \qquad (x_0)$$

$$
\begin{aligned}
0 \cdot y_1 + 1 \cdot y_2 - 1 \cdot y_3 + y_0 &\geq 0 \qquad (x_1) \\
-1 \cdot y_1 + 0 \cdot y_2 + 1 \cdot y_3 + y_0 &\geq 0 \qquad (x_2) \\
+1 \cdot y_1 - 1 \cdot y_2 + 0 \cdot y_3 + y_0 &\geq 0 \qquad (x_3)
\end{aligned}
$$

$$y_1, y_2, y_3 \geq 0 \qquad \text{(inequalities in primal)}$$

This is exactly the LP for min-max!

- Since primal is bounded and feasible, there is an optimum.

Hence primal-optimum = dual-optimum, i.e. max min = min max!

In general:

Given game M

max-min LP:

maximize x_0

subject to:

$$x_0 \le M_{11}x_1 + M_{21}x_2 + \cdots + M_{m1}x_m$$
$$x_0 \le M_{12}x_1 + M_{22}x_2 + \cdots + M_{m2}x_m$$
$$\vdots$$
$$x_0 \le M_{1n}x_1 + M_{2n}x_2 + \cdots + M_{mn}x_m$$

$$x_1 + x_2 + \cdots + x_m = 1$$
$$x_1, x_2, \ldots, x_m \ge 0$$

min-max LP:

minimize y_0

subject to:

$$y_0 \ge M_{11}y_1 + M_{12}y_2 + \cdots + M_{1n}y_n$$
$$y_0 \ge M_{21}y_1 + M_{22}y_2 + \cdots + M_{2n}y_n$$
$$\vdots$$
$$y_0 \ge M_{m1}y_1 + M_{m2}y_2 + \cdots + M_{mn}y_n$$

$$y_1 + y_2 + \cdots + y_n = 1$$
$$y_1, y_2, \ldots, y_n \ge 0$$

These two LPs are duals of each other.

Minimax theorem: max-min = min-max over mixed strategies

There exist mixed strategies $\tilde{x}$ and $\tilde{y}$ for Max and Min s.t.:
- $\tilde{y}$ is the best response to $\tilde{x}$ and
- $\tilde{x}$ is the best response to $\tilde{y}$

There is a Nash equilibrium over mixed strategies for zero-sum games.
