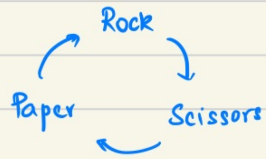
APPLICATIONS OF LP: ZERO-SUM GAMES - Part 2

Last video: max min, min max over pure strategies

This video: Mixed / randomized strategies.

Example:

$$\text{Max}\left\{\begin{array}{c|ccc} & \multicolumn{3}{c}{\overbrace{\text{Rock} \quad \text{Paper} \quad \text{Scissors}}^{\text{Min}}} \\ \text{Rock} & 0 & -1 & +1 \\ \text{Paper} & +1 & 0 & -1 \\ \text{Scissors} & -1 & +1 & 0 \end{array}\right. \qquad M_{RPS}$$



↳ Payoff for Maximizer

max - min - pure $(M_{RPS}) = -1$
min - max - pure $(M_{RPS}) = +1$

Consider a different strategy for Max:

- Max plays each strategy with $1/3$ Probability ← Mixed Strategy

$$\boxed{\sigma : \quad 1/3 \text{ Rock} + 1/3 \text{ Paper} + 1/3 \text{ Scissors}}$$

If Min plays pure strategy Rock, "Expected Payoff" $= 1/3 \cdot 0 + 1/3 \cdot 1 + 1/3 \cdot (-1)$
Paper, $= 1/3(-1) + 1/3(0) + 1/3 \cdot 1$
Scissors, $= 1/3 \cdot 1 + 1/3(-1) + 1/3(0)$

When Max plays mixed strategy σ and Min plays some pure strategy, the expected payoff $= 0$

- But Min can also play mixed strategies. Suppose Min plays:

$$\boxed{\tau : \quad \tfrac{1}{3}\text{ Rock} + \tfrac{1}{3}\text{ Paper} + \tfrac{1}{3}\text{ Scissors}}$$

- What is the expected payoff when σ and τ are played?

		Rock 1/3	Paper 1/3	Scissors 1/3
1/3	Rock	0	−1	+1
1/3	Paper	+1	0	−1
1/3	Scissors	−1	+1	0

$$\frac{1}{9}\left[\underbrace{(0 - 1 + 1)}_{\text{Row 1}} + \underbrace{(1 + 0 - 1)}_{\text{Row 2}}\underbrace{(-1 + 1 + 0)}_{\text{Row 3}}\right] = 0$$

Mixed Strategy: is a probability distribution over pure strategies.

Maximizer's pure strategies : $\{1, 2, 3, \dots, m\}$

Mixed strategy for Max : $x_1, x_2, \dots, x_m \in \mathbb{R}_{\geq 0}$

$$\text{s.t. } \sum_{i=1}^{m} x_i = 1$$

Minimizer's pure strategies : $\{1, 2, 3, \dots, n\}$

Mixed strategy for Min : $y_1, y_2, \dots y_n \in \mathbb{R}_{\geq 0}$

$$\text{s.t. } \sum_{j=1}^{n} y_j = 1$$

Payoff: Given $\sigma := (x_1, x_2, \dots, x_m)$ and $\tau := (y_1, y_2, \dots, y_n)$

$$\boxed{\text{Payoff}(\sigma,\tau) = \sum_{i \in \{1,\dots m\}} \sum_{j \in \{1 \dots n\}} x_i \cdot y_j \cdot m_{ij}}$$

$\hookrightarrow \; = x^T M y$

We are interested in the following quantities:

$$\max\min(M) = \max_{\substack{\text{mixed strategies } \sigma \\ \text{of Max}}} \; \min_{\substack{\text{mixed strategies } \tau \\ \text{of Min}}} \text{Payoff}(\sigma,\tau)$$

$$\min\max(M) = \min_{\substack{\text{mixed strategies } \tau \\ \text{of Min}}} \; \max_{\substack{\text{mixed strategies } \sigma \\ \text{of Max}}} \text{Payoff}(\sigma,\tau)$$

Notice that there are infinitely many mixed strategies.

Goal of this video: Given M, how do we compute

$\max\min(M)$?

- This does not look like a linear optimization problem since there is a mix of objectives and the cost that needs to be optimized: $x^T M y$ has bilinear terms

- We will now see that computing $\max\min(M)$ can be reduced to an LP problem.

1. Suppose we fix a strategy $\langle x_1, x_2, \dots, x_m \rangle$ for Max

- For example : $\langle 1/3, 1/3, 1/3 \rangle$ in M_{RPS}.

2. Given the strategy $\langle x_1, \dots x_m \rangle$ we want to find:

$$\min_{\text{mixed strategies of Min}} x^T M y$$

3. This can be written as a linear program (as x_i's are constants)

$$\boxed{\begin{array}{ll} \text{minimize} & x^T M y \\ \text{subject to} & y_1 + y_2 + \cdots + y_n = 1 \\ & y_1, y_2, \dots, y_n \geq 0 \end{array}} \quad (*)$$

For example, with $\langle 1/3, 1/3, 1/3 \rangle$ in RPS:

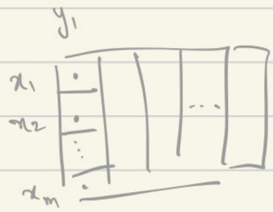
$$\begin{array}{ll} \text{minimize} & -\tfrac{1}{3} y_2 + \tfrac{1}{3} y_3 + \tfrac{1}{3} y_1 - \tfrac{1}{3} y_3 - \tfrac{1}{3} y_1 + \tfrac{1}{3} y_2 \\ \text{subject to} & y_1 + y_2 + y_3 = 1 \\ & y_1, \dots, y_3 \geq 0 \end{array}$$

4. The answer to above LP gives the best response of Min to Max's mixed strategy $\langle x_1, x_2, \dots, x_m \rangle$

Each $\langle x_1, x_2, \dots x_m \rangle \longrightarrow x_0$ (answer of above LP)

We want the maximum possible x_0.

5. Let us first write the dual of (*):



$$\boxed{\begin{aligned} &\text{minimize} && x^T M y \\ &\text{subject to} && y_1 + y_2 + \cdots + y_n = 1 \\ & && y_1, y_2, \ldots, y_n \geq 0 \end{aligned}}$$

min. $b^T y$
$A^T y = c$
$y \geq 0$

max $c^T x$ ←
$Ax \leq b$

maximize x_0

subject to:
$$x_0 \leq M_{11} x_1 + M_{21} x_2 + \cdots + M_{m1} x_m$$
$$x_0 \leq M_{12} x_1 + M_{22} x_2 + \cdots + M_{m2} x_m$$
$$\vdots$$
$$x_0 \leq M_{1n} x_1 + M_{2n} x_2 + \cdots + M_{mn} x_m$$

$$\boxed{\begin{aligned} &\text{maximize} && x_0 \\ &\text{subject to} && \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{n \times 1} x_0 \leq M^T x \end{aligned}}$$

n rows

(* *)

6. Primal is bounded (feasible region is contained in unit hypercube)

Hence: Optimum of primal = optimum of dual

- For a given strategy of Max $\langle x_1, x_2, \ldots, x_m \rangle$, the payoff obtained when Min plays her best response is given by the Optimum cost of LP (* *).

7. We want to maximize the optimum of LP (**) w.r.t. all mixed strategies of Maximizer.

- Hence we consider $\langle x_1, x_2, \dots, x_m \rangle$ as variables and add the constraint $\sum_{i=1}^{m} x_i = 1$ to (**)

$$\boxed{\begin{array}{ll} \text{maximize} & x_0 \\ \text{subject to} & \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} x_0 \leq M^T x \\ & x_1 + x_2 + \cdots + x_m = 1 \\ & x_1, x_2, \dots, x_m \geq 0 \end{array}}$$

LP for max min (M) [over mixed strategies]

Illustration on the RPS example:

maximize x_0

subject to:

$$x_0 \leq 0 \cdot x_1 + 1 \cdot x_2 - 1 \cdot x_3$$
$$x_0 \leq -1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3$$
$$x_0 \leq +1 \cdot x_1 - 1 \cdot x_2 + 0 \cdot x_3$$

$$x_1 + x_2 + x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

← LP to find max min.

	Rock	Paper	Scissors
Rock	0	-1	+1
Paper	+1	0	-1
Scissors	-1	+1	0