APPLICATIONS OF LINEAR PROGRAMMING: ZERO - SUM GAMES

REFERENCE: Section 8.1 of text:

Understanding and Using Linear Programming
— Matoušek & Gärtner

ZERO - SUM GAMES:

Two players: Maximizer Vs Minimizer

- Maximizer has m - strategies $\{1, 2, 3, \ldots, m\}$
- Minimizer has n - strategies $\{1, 2, 3, \ldots, n\}$
- An $m \times n$ payoff matrix M is given

Example:

Min chooses columns

Max chooses Rows

$$\begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline 1 & 10 & 0 & -1 \\ 2 & -2 & 4 & 0 \\ 3 & 5 & 3 & 1 \\ 4 & 7 & 2 & -2 \\ 5 & 4 & -1 & 1 \\ \hline \end{array} \longrightarrow M : \text{Payoff matrix}$$

- When Max plays i and Min plays j,
  Payoff $= m_{ij}$

$$\boxed{\text{Max receives } \quad m_{ij} \quad \text{from Min}}$$

OBJECTIVE / GOAL OF THE GAME:

- Maximizer wants to maximize the payoff
- Minimizer wants to minimize the payoff

Zero-sum: one's loss is the other's gain

- A payoff of 5 to Max is −5 to Min
- A payoff of −3 to Max is +3 to Min

NOTE: Many situations in economics / AI / finance involving strategic reasoning can be modeled as zero-sum games.

SOME EXAMPLES:

I)

	1	2	3
1	10	0	−1
2	−2	4	0
3	5	3	1
4	7	2	−2
5	4	−1	1

When Max plays 1, best strategy of Min is 3, Payoff = −1

Max plays	best strategy of Min	Payoff
1	3	−1
2	1	−2
3	3	1
4	3	−2
5	2	−1

$\boxed{\text{max min} = 1}$

When Min plays 1, best strategy of Max is 1, Payoff = 10

Min plays	best strategy of Max	Payoff
1	1	10
2	2	4
3	3/5	1

$\boxed{\text{min max} = 1}$

II)

	1	2	3
1	10	0	-1
2	-2	4	0
3	5	3	7
4	7	2	-2
5	4	-1	1

When Max plays 1, best strategy of Min is 3, Payoff = -1
2 — 1 — -2
3 — 2 — 3
4 — 3 — -2
5 — 2 — -1

$\boxed{\text{max min} = 3}$

When Min plays 1, best strategy of Max is 1, Payoff = 10
2 — 2 — 4
3 — 3 — 7

$\boxed{\text{min max} = 4}$

Remarks:

max min:
- Max plays first.
- Knowing Max's strategy Min gives her best response
- Knowing that Min will play best response, Max plays a strategy that maximizes the payoff.

min max
- Min plays first
- Max gives her best response
- Min plays a strategy that minimizes the payoff, knowing that Max will play best response.

More formally:

Given a game represented by payoff matrix M:

$$\text{max-min-pure}(M) = \max_{i \in \{1,\dots,m\}} \ \min_{j \in \{1,2,\dots,n\}} m_{ij}$$

$$\text{min-max-pure}(M) = \min_{j \in \{1,\dots,n\}} \ \max_{i \in \{1,\dots,m\}} m_{ij}$$

Pure strategies:
- The choices $1,\dots m$ for Max are called her pure / deterministic strategies
- Choices $1\dots n$ of Min are called her pure / deterministic strategies

Later we will see other kinds of strategies

Lemma: $\text{max-min-pure}(M) \leq \text{min-max-pure}(M) \quad \forall M$

Proof: For each $i \in \{1,2,\dots m\}$ (a pure strategy of Max):

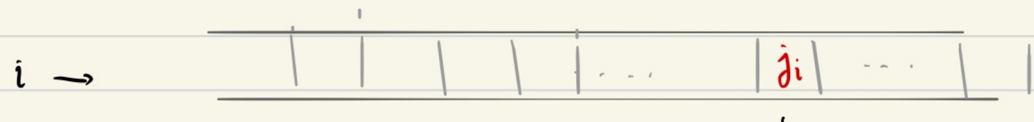
$$\min_{j \in \{1,2,\dots n\}} m_{ij} \leq \min_j \max_i m_{ij}$$

Hence $\max_i \min_j m_{ij} \leq \min_j \max_i m_{ij}$ $\square$

Illustration of this fact:

For each $i \in \{1, 2, \dots m\}$ (a pure strategy of Max):

$$\min_{j \in \{1,2,\dots n\}} m_{ij} \le \min_j \max_i m_{ij}$$



Now: $m_{ij_i} \le m_{i1} \le \max_i m_{i1}$

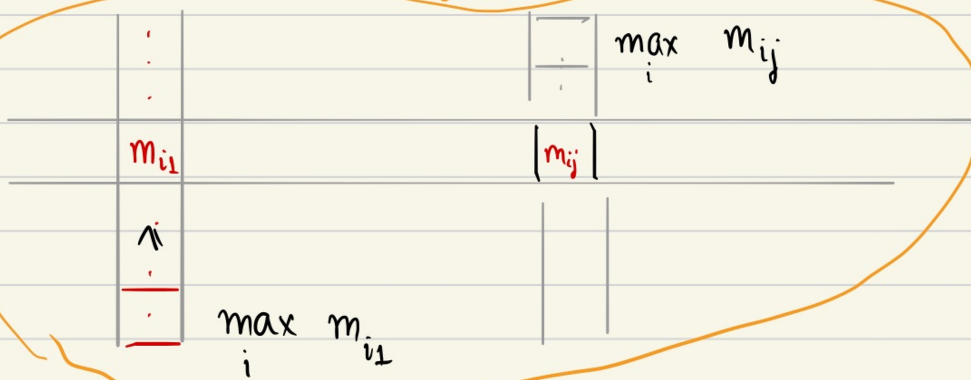
$\le m_{i2} \le \max_i m_{i2}$

$\vdots$

$\le m_{in} \le \max_i m_{in}$

→ $m_{ij_i} \le$ least among these numbers

$\Rightarrow m_{ij_i} \le \min_j \max_i m_{ij}$



Hence: $m_{ij_i} \le \min_j \max_i m_{ij}$

Saddle points:

We saw that max-min-pure(M) $\leq$ min-max-pure(M)

In some cases, max-min-pure = min-max-pure

(3, 3)

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 10 & 0 & -1 \\ 2 & -2 & 4 & 0 \\ 3 & 5 & 3 & 1 \\ 4 & 7 & 2 & -2 \\ 5 & 4 & -1 & 1 \end{array} \longrightarrow$$

Least in row

Greatest in column

m_{kl} is a saddle point if $m_{kl} = \min_j m_{kj}$

$= \max_i m_{il}$

$$\boxed{\text{- max-min-pure} = \text{min-max-pure iff there is a saddle point.}}$$

Prove this statement: Exercise.

NASH EQUILIBRIUM (over pure strategies):

When max-min-pure (M) = min-max-pure (M)

the game is said to have a **Nash equilibrium over pure strategies**

- In this case, there exist strategies i, j for Max / Min s.t.

i) j is the best response of Min to i

and

ii) i is the best response of Max to j

	1	2	3
1	10	0	−1
2	−2	4	0
3	5	3	1
4	7	2	−2
5	4	−1	1

$\longrightarrow$ Least in row

Greatest in column

Nash equilibrium strategies: 3 for Max

3 for Min

- Max does not gain anything by deviating
- Similarly, Min does not have an incentive to deviate.

Summary:

- 1. Zero-sum games, Pure strategies
- 2. max-min-pure and min-max-pure
- 3. In general: max-min-pure $\leq$ min-max-pure
- 4. Nash equilibrium: when max-min-pure = min-max-pure.

Where is Linear Programming in all this?

- Next video: different kind of strategies,
  - ↳ max min and min max over such strategies
  - ↳ significant use of LP.