DUAL OF AN LP

GOAL: Given an LP, associate another LP to it (called its "dual") which has interesting relations to the original LP

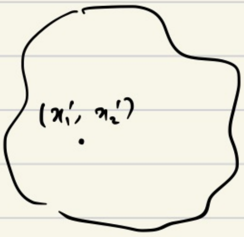
REFERENCE: Section 6.1 of text:

Understanding and Using Linear Programming
— Matoušek & Gärtner

Intuition:

$$\text{maximize} \quad 2x_1 + 3x_2$$

$$\begin{aligned} \text{subject to} \quad 4x_1 + 8x_2 &\leq 12 \\ 2x_1 + x_2 &\leq 3 \\ 3x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

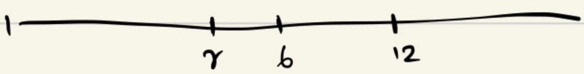


$$2x_1' + 3x_2'$$

$$2x_1' \leq 4x_1' \qquad (\text{since } x_1' \geq 0)$$
$$3x_2' \leq 8x_2' \qquad (\text{since } x_2' \geq 0)$$

$$2x_1' + 3x_2' \leq 4x_1' + 8x_2' \leq 12 \quad (\text{from LP constraints})$$

Suppose r is the optimum of the above LP.



: $2x_1 + 4x_2 \leq b$

So the cost: $2x_1 + 3x_2 \leq b$

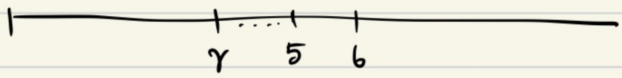
maximize $2x_1 + 3x_2$

subject to

$$\left.\begin{aligned} 4x_1 + 8x_2 &\le 12 \\ 2x_1 + x_2 &\le 3 \end{aligned}\right\}$$
$$3x_1 + 2x_2 \le 4$$
$$x_1 , x_2 \ge 0$$

add: $\frac{1}{3}(6x_1 + 9x_2) \le 15$

$2x_1 + 3x_2 \le 5$



+ive value

$$\left\{\begin{aligned} y_1 &\times 4x_1 + 8x_2 \le 12 \\ y_2 &\times 2x_1 + x_2 \le 3 \\ y_3 &\times 3x_1 + 2x_2 \le 4 \end{aligned}\right.$$

$$(4y_1 + 2y_2 + 3y_3)x_1 + (8y_1 + y_2 + 2y_3)x_3 \le 12y_1 + 3y_2 + 4y_3$$

≥ 2 $\qquad$ ≥ 3

Minimize $12y_1 + 3y_2 + 4y_3$

$4y_1 + 2y_2 + 3y_3 \ge 2$

$8y_1 + y_2 + 2y_3 \ge 3$

$y_1, y_2, y_3 \ge 0$

$\underbrace{\gamma \le 12y_1 + 3y_2 + 4y_3}$

→ Dual of the LP that we started with

→ optimum (dual) ≥ optimum (original LP)

maximize $c^T x$	minimize $b^T y$
subject to $Ax \leq b$	subject to $A^T y \geq c$
$x \geq 0$	$y \geq 0$
Primal	Dual

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$\begin{array}{c} y_1 \times \\ y_2 \times \\ \vdots \\ y_m \times \end{array} \begin{bmatrix} - \; - \; - \; - \; - \\ \\ A : m \times n \\ \\ \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \leq \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\downarrow$$

Objective: Minimize $b^T y$

Constraints: $A^T y \geq c$

$y \geq 0$

Weak Duality: (Proposition 6.1.1 in the text)

Primal	Dual
max. $c^T x$	min. $b^T y$
subj to	subj to
$Ax \le b$	$A^T y \ge c$
$x \ge 0$	$y \ge 0$

$c^T x \le b^T y$

x' $\quad$ y'

For every x', y' s.t. x' is a feasible soln. of primal and y' is a feasible soln. of dual.

Corollary:

1) If primal is unbounded, dual is infeasible.

Claim: If dual is unbounded, primal is infeasible.

Question: Can this claim be proved using the above weak duality theorem?

Duality Theorem:

Primal opt.

r

= dual optimum

Exactly one of the foll. possibilities occur:

- 1. Both Primal (P) and Dual (D) are infeasible
- 2. Primal is unbounded, dual is infeasible
- 3. (P) is infeasible, dual is unbounded
- 4. Both primal and dual have an optimum. In this case, both the optima coincide.

Exercise: Write the dual for the following LPs.

1) maximize: $12x_1 + 3x_2 + 5x_3$

$$7x_1 + 4x_2 + 2x_3 \leq 100$$

$$2x_1 + 7x_3 \leq 50$$

$$x_1, x_2, x_3 \geq 0$$

2) maximize $2x_1 + 3x_2$

subject to:

$$4x_1 + 8x_2 \geq 12$$

$$2x_1 + x_2 \leq 3$$

$$3x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$