Problem Set 2
Linear Optimization 2020

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January 18, 2020

Problem 1. Input to the problem is a network consisting of a cycle on n nodes numbered 0 to $n-1$ clockwise, and a set of calls C. Each call in C is a pair (i, j) where the call originates at node i and destined for node j. The call can be routed clockwise or counter-clockwise in the cycle. The load L_i on the edge $(i, i+1)$ is the number of calls routed through the edge ($i+1$ is taken modulo n). The goal to minimize the maximum load in the network i.e. $\max_{0 \leq i \leq n} L_i$.

Give an LP for this problem.
Hint: Define a variable denoting the max load.

Problem 2. An independent set in a graph $G = (V, E)$ is a subset $C \subseteq V$ such that there is no edge among any pair of vertices in C. The objective is to choose the largest independent set in G.

Give an ILP formulation of the problem.

Problem 3. Consider the integer linear program

$$\begin{array}{rcl} \max & & c'x \\ Ax & \leq & b \\ x & \geq & 0 \\ x & \in & \mathbb{Z} \end{array}$$

where A, b, and c are all composed of positive integers. Let LP be the linear program obtained by relaxing the constraint that x be integer. Call the solution to LP x. Show that

i. ILP also has a solution x'

ii. its cost can be no farther from optimal than $\Sigma_{i=1}^{n} c_i$.

Problem 4. Suppose you have an LP formulation with n unconstrainted variables. Show that you can reformulate the LP with $n+1$ nonnegative variables.

Problem 5. Show that a basis for a subspace can be extended to a basis for the whole vector space.

Problem 6. A universe D consisting of finite number of elements, and a family $S_1, S_2, \ldots, S_m$ of sets, with each $S_i \subseteq D$. Assume that $\cup_{i \in \{1,2,\ldots,m\}} S_i = D$. The objective is to find a minimum size subset $W \subseteq \{1, \ldots, m\}$ such that $\cup_{i \in W} S_i = D$.

Write the ILP and the relaxed LP for the problem.

Problem 7. Consider the feasible region for the following set of inequalities.

$$
\begin{aligned}
x + 2y &\leq 6 \\
x + y &\leq 4 \\
x &\geq 0 \\
y &\geq 0 \\
x, y &\in \mathbb{Z}
\end{aligned}
$$

Prove that for any linear objective function, the ILP optimum coincides with the optimum of the LP relaxation of the problem.
Hint: Requires extreme thinking.

Compiled from various sources, by Ashwani Anand (M.Sc. CS, Ist year) and Sricharan AR (M.Sc. CS, Ist year)