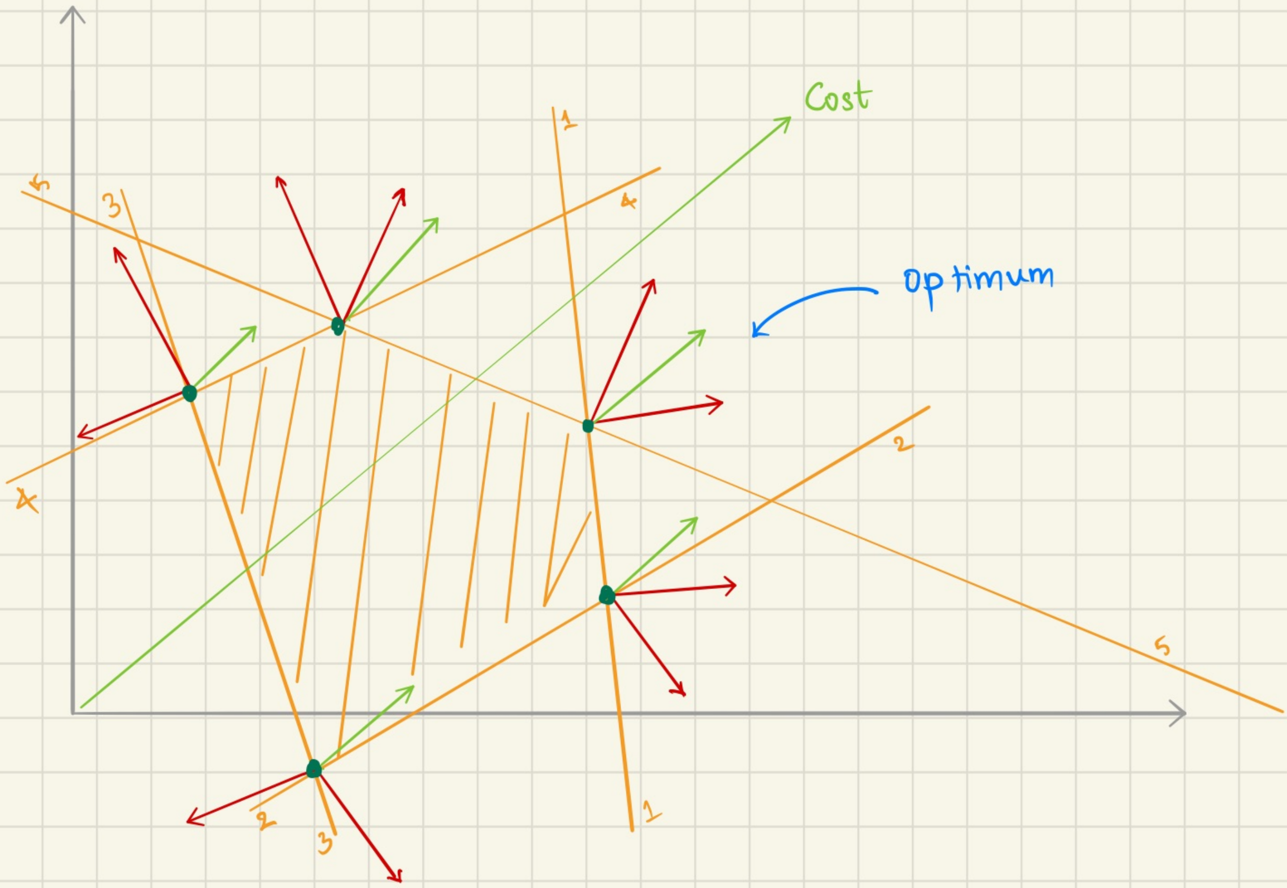
GEOMETRIC INTERPRETATION OF DUALITY - Part 2



Given LP maximize $c^T x$ subject to $Ax \leq b$

– non-degenerate, has optimum

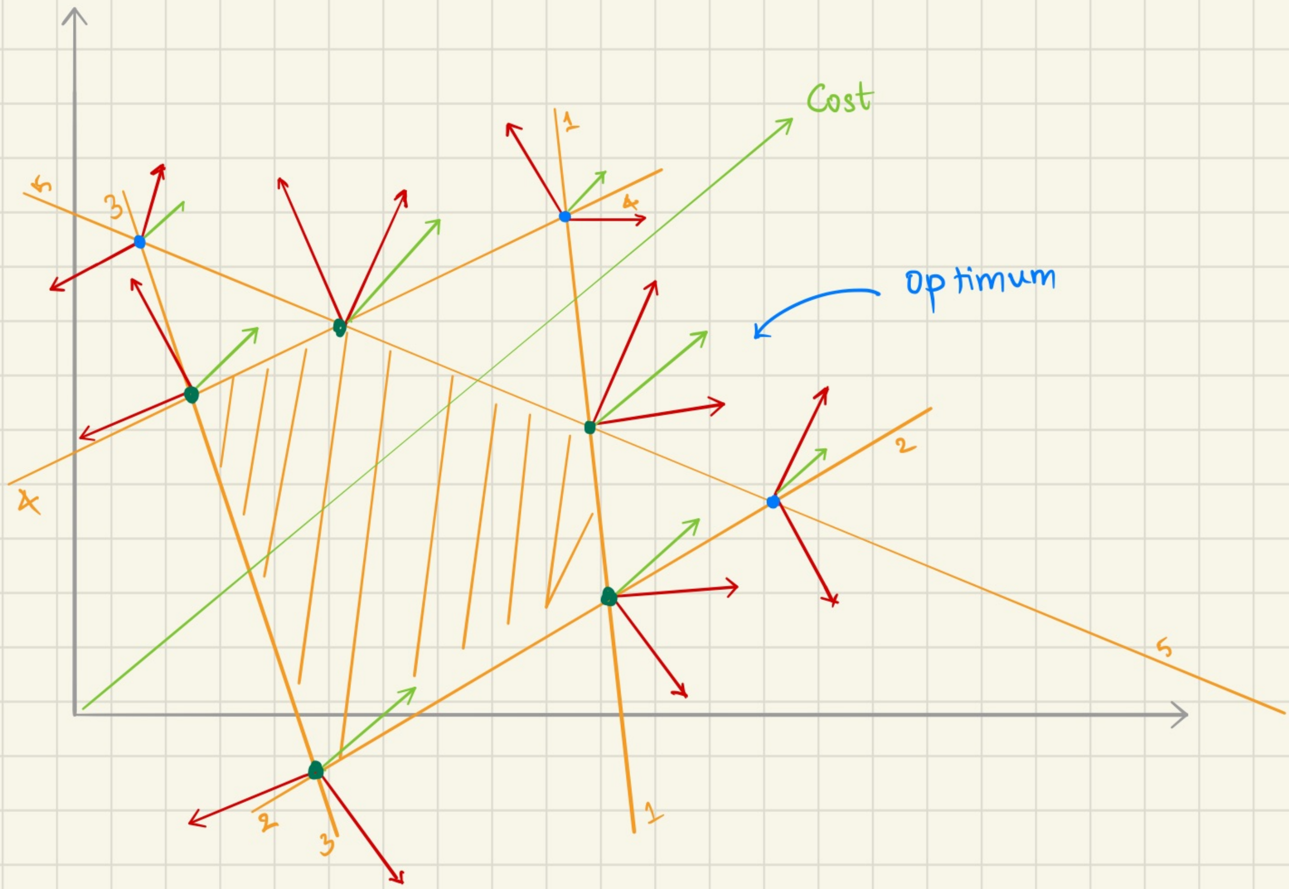
Intersection points: Points satisfying

$$\begin{aligned} A_{i_1} x &= b_{i_1} \\ A_{i_2} x &= b_{i_2} \\ &\vdots \\ A_{i_n} x &= b_{i_n} \end{aligned}$$

for some n linearly independent rows from $Ax \leq b$

– These points need not satisfy the other inequalities.

Extreme points: Intersection points that are also feasible.



Define $F :=$ set of intersection points such that cost vector 'c' is a non-negative combination of the outward normals of the defining hyperplanes

Point x satisfying $A_{i_1} x = b_1, \dots, A_{i_n} x = b_n$ $\in F$ if $c^T = \alpha_1 A_{i_1} + \alpha_2 A_{i_2} + \dots + \alpha_n A_{i_n}$

such that $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_n} \geq 0$

Theorem: Optimum of LP is a point in F with minimum cost.

To Prove: For every point $x \in F$, $c^T x \geqslant c^T x_0$, where x_0 is the optimum of the LP.

Proof: To show: $c^T(x - x_0) \geqslant 0$

Since $x \in F$: $c^T = \alpha_{i_1} A_{i_1} + \alpha_{i_2} A_{i_2} + \dots + \alpha_{i_n} A_{i_n}$, $\alpha_{i_1} \dots \alpha_{i_n} \geqslant 0$

$$c^T(x - x_0) = \alpha_{i_1} A_{i_1}(x - x_0) + \alpha_{i_2} A_{i_2}(x - x_0) + \dots + \alpha_{i_n} A_{i_n}(x - x_0)$$

$A_{i_j}(x) = b_{i_j}$ $\quad A_{i_j}(x_0) \leqslant b_{i_j} \Rightarrow A_{i_j}(x - x_0) \geqslant 0$

$\Rightarrow c^T(x - x_0) \geqslant 0$

- For $x \in F$, with $A_{i_1} x = b_{i_1}, \dots A_{i_n} x = b_{i_n}$

suppose $c^T = \alpha_{i_1} A_{i_1} + \alpha_{i_2} A_{i_2} + \dots + \alpha_{i_n} A_{i_n}$, $\alpha_{i_1} \dots \alpha_{i_n} \geqslant 0$

define $y \in \mathbb{R}^m$: $\langle y_1, y_2, \dots, y_m \rangle$

$y_{i_1} = \alpha_{i_1}$

$\vdots$ $\quad y_j = 0$ for $j \notin \{\alpha_{i_1} \dots \alpha_{i_n}\}$

$y_{i_n} = \alpha_{i_n}$

Then: $\boxed{c^T x = b^T y}$

define $\overline{F} = \{ \langle y_1, y_2, \dots y_m \rangle$ as above $|$ for each $x \in F \}$

Lemma: $\min\{ b^T y \mid y \in \overline{F}\} = \min\{c^T x \mid x \in F\} =$ optimum of primal

MOTIVATION FOR THE DUAL:

Now consider the LP:

$$\boxed{\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to:} & A^T y = c \\ & y \geq 0 \end{array}}$$

$Ax \leq b$

$y_1 x$ — A_1

$\vdots$

$y_m x$ — A_m

cost is a non-negative combination of hyperplanes

→ Above LP is the dual of our considered primal.

Theorem: {Extreme points of dual} equals $\bar{F}$

Assuming this theorem: + fact that $c^T x_0 \leq b^T y$ $\forall y$ feasible (dual)

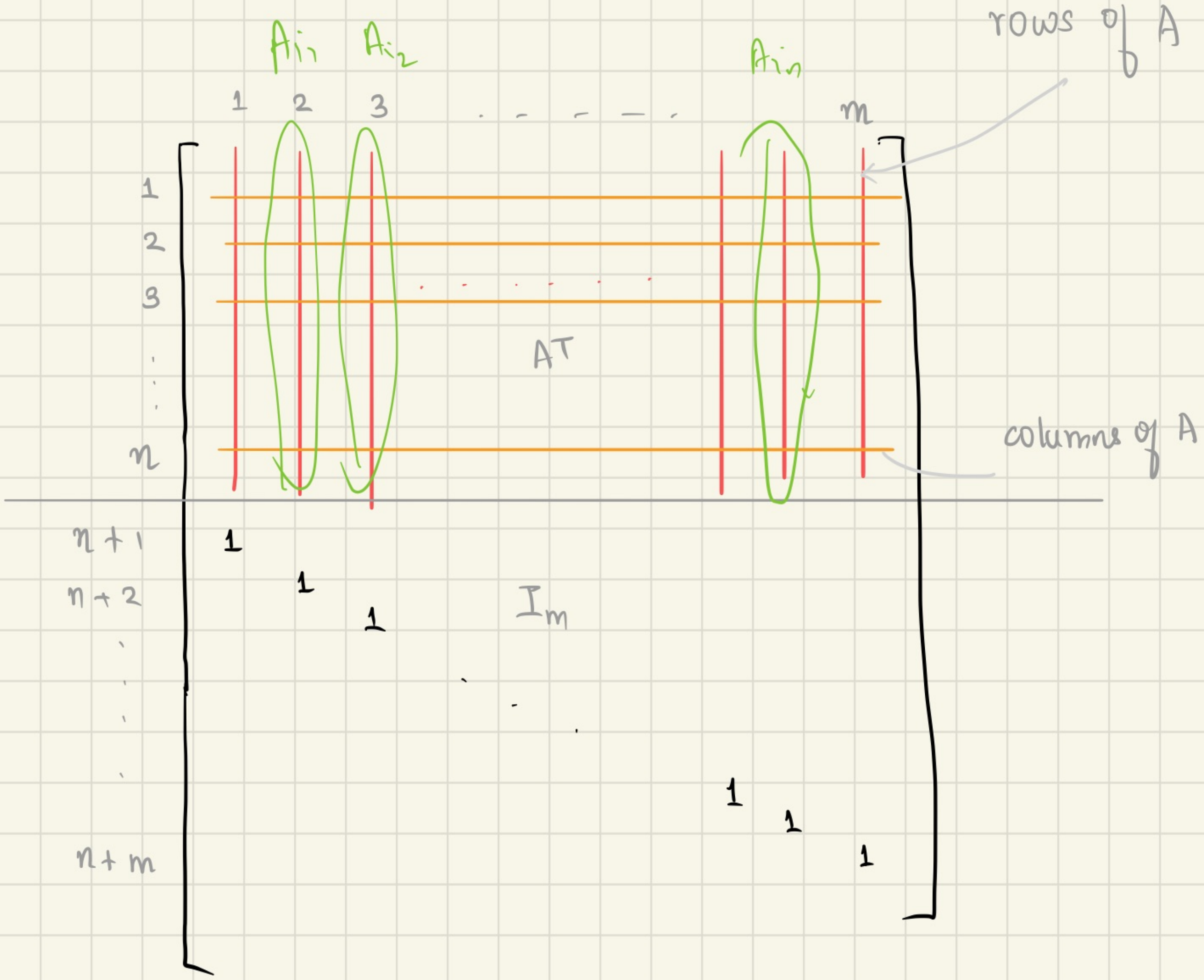
$$\text{Optimum (dual)} = \min \{ b^T y \mid y \in \bar{F} \} = \text{optimum (Primal)}$$

Constraint matrix of $A^T y = c$, $y \geq 0$:

$$
\begin{array}{c|c}
 & 1 \quad 2 \quad 3 \quad \cdots \quad m \\
\hline
\begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{array} & \\
\begin{array}{c} n+1 \\ n+2 \\ \vdots \\ n+m \end{array} &
\end{array}
\begin{bmatrix}
A^T \\
\hline
I_m
\end{bmatrix}
\qquad
I_m = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}
$$

rows of A

columns of A



Claim: First 'n' rows of above matrix are linearly independent.

Proof: Since primal has an optimum (extreme point), there are n rows of A: $A_{i_1}, A_{i_2}, \ldots, A_{i_n}$ which are independent.

Let
$$Z = \begin{bmatrix} a_{i_1 1} & a_{i_1 2} & \cdots & a_{i_1 n} \\ a_{i_2 1} & a_{i_2 2} & \cdots & a_{i_2 n} \\ & \vdots & & \\ a_{i_n 1} & a_{i_n 2} & \cdots & a_{i_n n} \end{bmatrix}$$

Row-rank(Z) = n = Col-rank(Z)

$\Rightarrow$ Columns of Z are independent $\Rightarrow$ required claim.

Extreme Points (dual) $\subseteq \bar{F}$:

At an extreme point $\bar{y}$ of dual, we need equality for 'm' linearly independent constraints.

- We already have equality at the first 'n' constraints.
- So, we will have $m-n$ of y_i to be 0.

$\therefore$ Let :

$$\boxed{\begin{array}{ll} y_{j_1}, y_{j_2}, \dots, y_{j_{m-n}} & = 0 \\ y_{i_1}, y_{i_2}, \dots, y_{i_n} & \geq 0 \end{array}}$$

- Rows $1, 2, \dots n, \; n+j_1, n+j_2, \dots n+j_{n-m}$ of constraint matrix will be linearly independent by definition.

We now have:

$$c^T = y_{i_1} A_{i_1} + y_{i_2} A_{i_2} + \dots + y_{i_n} A_{i_n}$$

$\rightarrow$ c is a non-negative combination of n rows of A

$\Rightarrow$ $\bar{y} \in \bar{F}$

$\bar{F} \subseteq$ Extreme points (dual)

$y \in \bar{F}$: $\quad y_{i_1}, y_{i_2}, \dots \; y_{i_n} \geq 0$

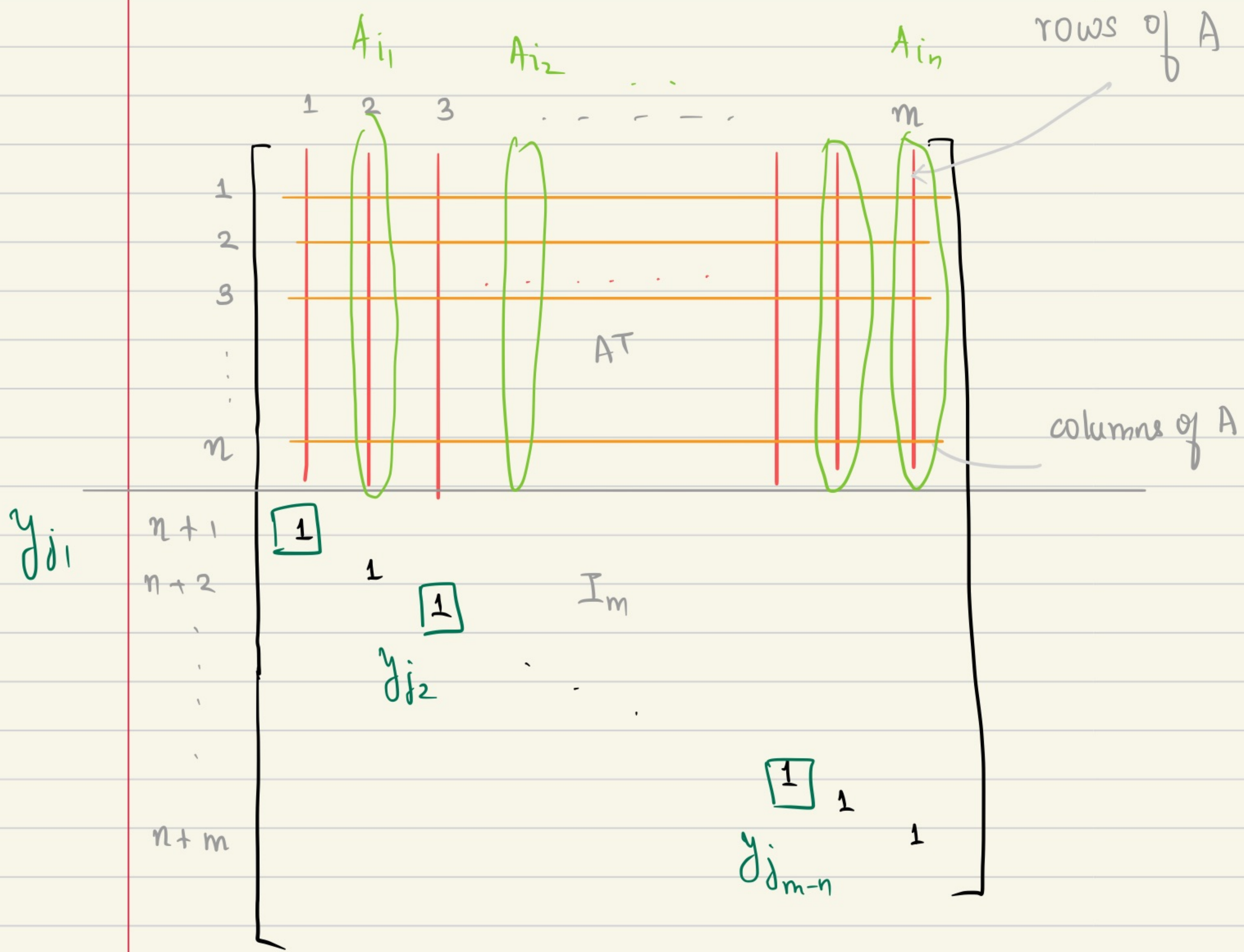
$$y_{j_1}, y_{j_2}, \dots \; y_{j_{n-m}} = 0$$

Moreover: $\quad c^T = y_{i_1} A_{i_1} + y_{i_2} A_{i_2} + \dots + y_{i_n} A_{i_n}$

and $A_{i_1}, A_{i_2}, \dots, A_{i_n}$ are linearly independent

This will imply that, in the constraint matrix:

rows $1, 2, \dots, n, \; n+j_1, n+j_2, \dots, n+j_{m-n}$ are linearly independent.



Note that if some non-zero combination of rows $1, 2, \ldots n, n+j_1, n+j_2, \ldots n+j_{m-n}$ is $\bar{0}$,

then the combination restricted to first 'n' rows is also $\bar{0}$.

→ This contradicts that the first n rows are independent.

Summary:

- Consider all intersection points of primal such that cost is a non-negative combination of the rows corresponding to the defining hyperplanes.

- Optimum (primal) is a point with least cost in the above set.

- Leads to the definition of dual, and duality theorem.