A GEOMETRIC INTERPRETATION OF DUALITY → Part 1

REFERENCE: Lecture 11 from Prof. Sundar Vishwanathan's notes on Linear Optimization

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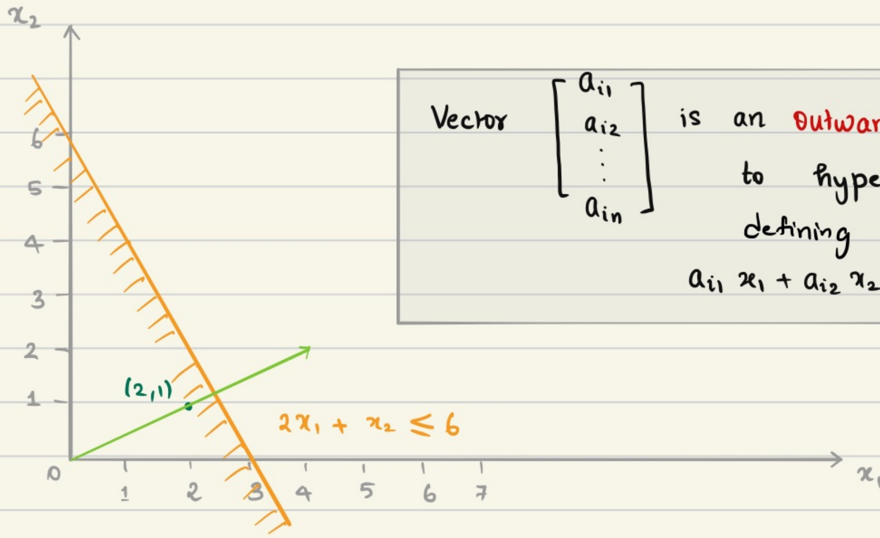
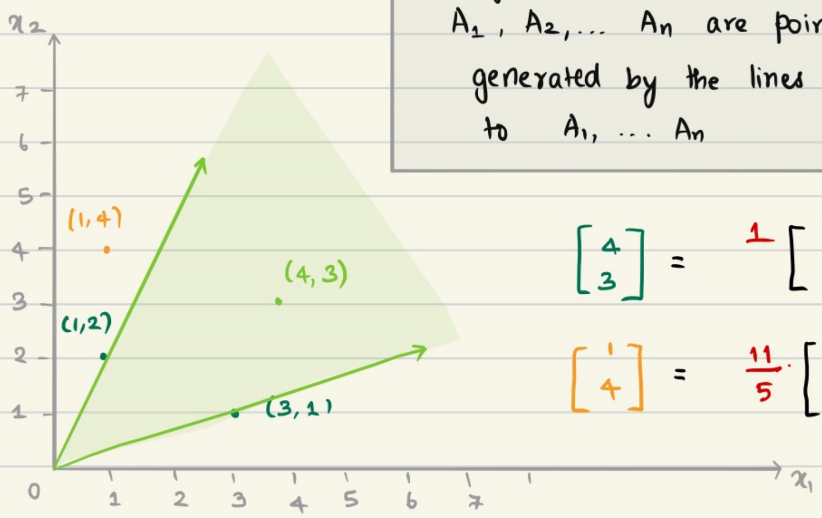
Some basic observations:

1)



Vector $\begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}$ is an outward normal to hyperplane defining the half-space $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i$

2)



Non-negative combinations of vectors $A_1, A_2, \dots A_n$ are points in the cone generated by the lines joining origin to $A_1, \dots A_n$

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \frac{11}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{2}{5} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Consider an LP: $\boxed{\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b \\ & x \text{ unrestricted} \end{array}}$

$A: m \times n$
$c: n \times 1$
$x: n \times 1$
$b: m \times 1$

Assume that the LP is non-degenerate, and has an optimum

Extreme point: a feasible solution that:

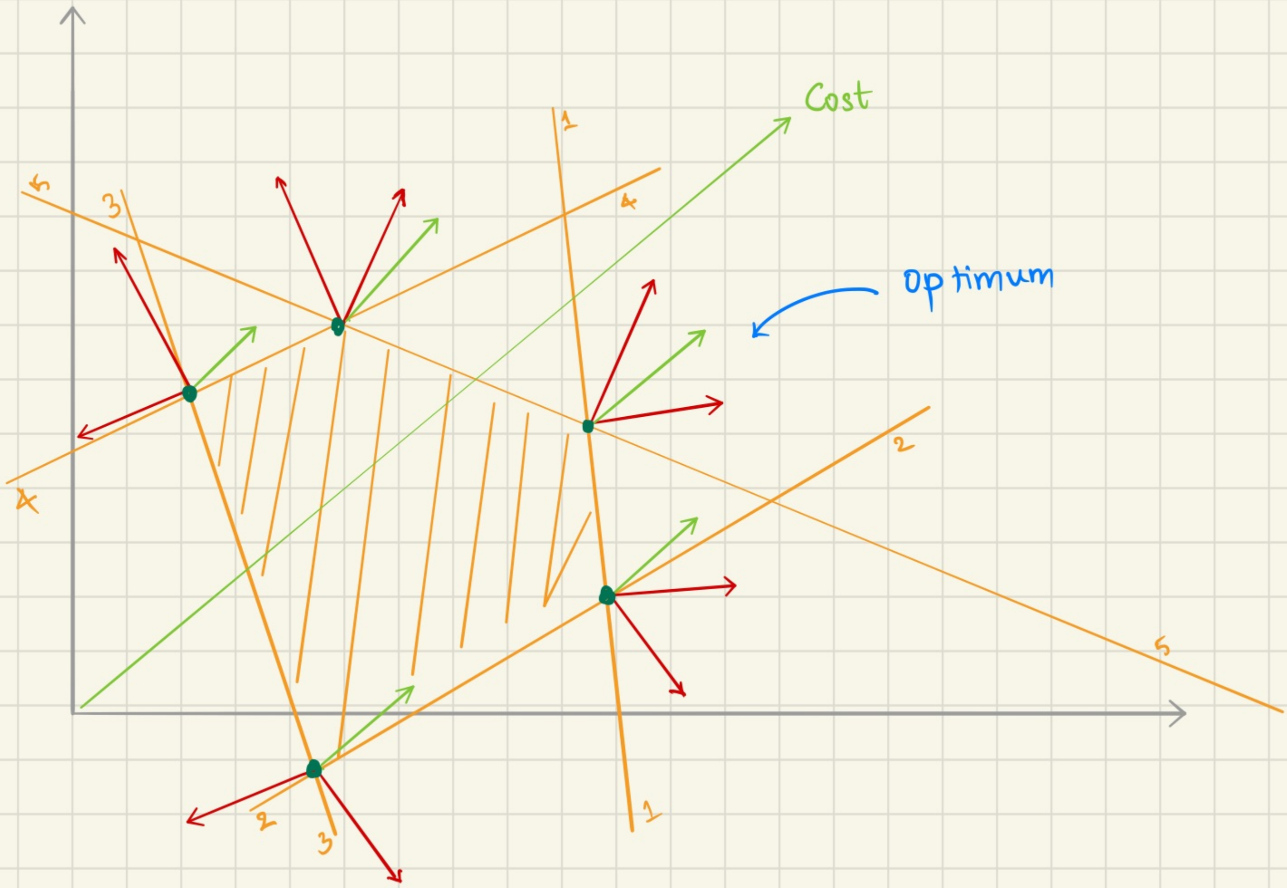
- is obtained as the intersection of n linearly independent hyperplanes from $A_1 x = b_1, A_2 x = b_2, \ldots, A_m x = b_m$

$A'x = b'$ (the set of n hyperplanes)
$A''x < b''$ (others)

Theorem: An extreme point defined by $A'x = b'$, $A''x < b''$ is an optimum

iff

the cost vector 'c' can be written as a non-negative combination of $A'_1, A'_2, \ldots A'_n$



Theorem: An extreme point defined by $A'x = b'$, $A''x < b''$ is an optimum iff the cost vector 'c' can be written as a non-negative combination of $A'_1, A'_2, \ldots A'_n$

KEY OBSERVATION

At optimum extreme point, cost decreases along all directions to the neighbouring extreme points.

CHARACTERIZING NEIGHBOURING DIRECTIONS FROM AN EXTREME POINT:

u_0 : extreme point

$$A'x = b'$$
$$A''x < b''$$

Suppose u_i is a point such that: $A'_j u_i = b_j \quad \forall j \neq i$

$$A'_i u_i < b_i$$

For example: u_1 :

$$A'_1 u_1 < b_1$$
$$\left.\begin{array}{l} A'_2 u_2 = b_2 \\ \vdots \\ A'_n u_n = b_n \end{array}\right\} \text{tight}$$

- From each extreme point, there are 'n' rays moving out to its neighbours.

Vectors $u_1 - u_0 \quad u_2 - u_0 \quad \cdots \quad u_n - u_0$ characterize these directions.

- Note that:

$$A'(u_1 - u_0) = \begin{bmatrix} A'_1 u_1 - A'_1 u_0 \\ A'_2 u_1 - A'_2 u_0 \\ \vdots \\ A'_n u_1 - A'_n u_0 \end{bmatrix} = \begin{bmatrix} \text{some negative value} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- By sufficient scaling, we can assume that $A'_1 (u_1 - u_0) = -1$

Similarly $A'_i (u_i - u_0) = -1$ $\qquad A'(u_i - u_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th}$ place

We have now 'n' vectors:

$$U = \begin{bmatrix} u_1 - u_0 & u_2 - u_0 & \cdots & u_n - u_0 \end{bmatrix}$$

s.t.

$$A' U = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & 0 & & \vdots \\ 0 & 0 & & -1 \end{bmatrix}$$

This implies:

$$\boxed{U = -(A')^{-1}}$$

Theorem: Directions to neighbours from an extreme point defined by $A'x = b'$, $A''x < b''$ are given by columns of $-(A')^{-1}$.

At optimum, cost decreases along all its neighbours.

$$c^T (u_i - u_0) \leq 0$$

$$\therefore \quad c^T \underbrace{\left[-(A')^{-1}\right]_i}_{i^{th} \text{ column of } -(A')^{-1}} \leq 0 \quad \forall i \in \{1 .. n\}$$

$$\therefore \quad \boxed{c^T (A')^{-1}_i \geq 0 \quad \forall i \in \{1 \dots n\}} \quad (*)$$

Writing the cost vector as a linear combination of rows of A':

$A'x = b'$ contains 'n' linearly independent hyperplanes:

$$A_1' x = b_1'$$
$$\vdots$$
$$A_n' x = b_n'$$

- Vectors: $A_1', \dots A_n'$ are linearly independent.

  $\therefore$ Cost c can be written as a linear combination of them

$$\underset{\sim}{c} = \alpha_1 . \underset{\sim}{A_1'} + \alpha_2 \underset{\sim}{A_2'} + \dots + \alpha_n \underset{\sim}{A_n'}$$

$n \times 1$: column vectors.

Rewriting: $$c^T = \alpha_1 A_1'^T + \alpha_2 A_2'^T + \dots + \alpha_n A_n'^T$$

- Note that: $$c^T . (A')^{-1}_1 = \alpha_1$$
$$\vdots$$
$$c^T . (A')^{-1}_n = \alpha_n$$

$$A' (A')^{-1} = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{bmatrix}$$

From $(*)$, $\alpha_1, \alpha_2, \dots \alpha_n \geq 0$

- Proves that at optimum, cost can be written as a non-negative linear combination of the outward normals of the 'n' hyperplanes defining the extreme point!