COMPLEMENTARY SLACKNESS

maximize $c^T x$	minimize $b^T y$
subj. to $Ax \leq b$	subj. to $A^T y = c$
	$y \geq 0$

Theorem: Let x_0, y_0 be feasible solutions of primal and dual respectively.

Then: x_0 and y_0 are optima $\Leftrightarrow$ $c^T x_0 = b^T y_0$

Proof: $(\Rightarrow)$: duality

$(\Leftarrow)$: Suppose $c^T x_0 = b^T y_0$.

1) Both primal and dual are feasible. Hence: both have optima.

2) $c^T x \leq b^T y_0$ $\forall$ feasible solutions of primal (weak duality)

Since $c^T x_0 = b^T y_0$, x_0 is the optimum of the primal.

3) optimum cost of dual = Optimum cost of primal.

Again, as $b^T y_0 = c^T x_0$, y_0 is the optimum of dual.

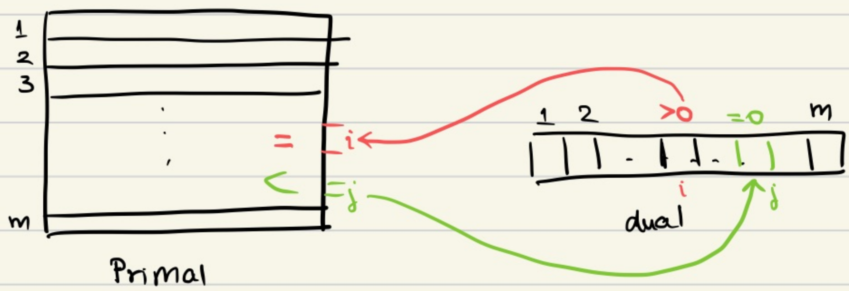
$$\boxed{\begin{array}{l}\text{maximize} \quad c^T x \\ \text{subj. to} \quad Ax \leq b\end{array}} \qquad \boxed{\begin{array}{l}\text{minimize} \quad b^T y \\ \text{subj. to} \quad A^T y = c \\ \qquad\qquad y \geq 0\end{array}}$$

Theorem (COMPLEMENTARY SLACKNESS CONDITION):

x_0: primal feasible solution $\qquad$ y_0: dual feasible solution

Then: $\boxed{c^T x_0 = b^T y_0 \text{ iff } (y_0)_i > 0 \Rightarrow A_i x_0 = b \quad \forall i \in \{1, 2, \ldots, m\}}$

dual variable slack	$\Rightarrow$	Primal inequality tight
Primal inequality slack	$\Rightarrow$	dual variable tight



- Gives another criteria to check for optimum.

Proof of complementary slackness theorem:

($\Leftarrow$:) Suppose $(y_0)_i > 0 \Rightarrow A_i x_0 = b_i$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}$$

To show: $c^T x_0 = b^T y_0$

We know:

$$c_1 = a_{11}(y_0)_1 + a_{21}(y_0)_2 + \cdots + a_{m1}(y_0)_m$$
$$c_2 = a_{12}(y_0)_1 + a_{22}(y_0)_2 + \cdots + a_{m2}(y_0)_m$$
$$\vdots$$
$$c_n = a_{1n}(y_0)_1 + a_{2n}(y_0)_2 + \cdots + a_{mn}(y_0)_m$$

$$c^T x_0 = c_1(x_0)_1 + c_2(x_0)_2 + \cdots + c_n(x_0)_n$$
$$= (y_0)_1 A_1 x_0 + (y_0)_2 A_2 x_0 + \cdots + (y_0)_m A_m x_0$$

By hypothesis, whenever $(y_0)_i > 0$, $A_i x_0 = b_i$

$\therefore$ $c^T x_0 = b^T y_0$

($\Rightarrow$:) Suppose $c^T x_0 = b^T y_0$

To show: $(y_0)_i > 0 \Rightarrow A_i x_0 = b_i$

Once again use the fact that:

$$c^T x_0 = (y_0)_1 A_1 x_0 + (y_0)_2 A_2 x_0 + \cdots + (y_0)_m A_m x_0$$

If for some i with $(y_0)_i > 0$, we have $A_i x < b_i$

then $c^T x_0 < b^T y_0$ (contradicting our hypothesis).

$\Rightarrow$ whenever $(y_0)_i > 0$, we have $A_i x = b_i$

COMPLEMENTARY SLACKNESS FOR OTHER PRIMAL - DUAL PAIRS

Primal	Dual
maximize $c^T x$	minimize $b^T y$
subj. to $Ax \leq b$	subj. to $A^T y \geq c$
$x \geq 0$	$y \geq 0$

Theorem: Let x_0, y_0 be feasible solutions of primal and dual respectively.

Then: $c^T x_0 = b^T y_0$ iff

i) $(y_0)_i > 0 \Rightarrow A_i x_0 = b_i \quad \forall i \in \{1, 2, \dots, m\}$

AND

ii) $(x_0)_j > 0 \Rightarrow A_j^T y_0 = c_j \quad \forall j \in \{1, 2, \dots, n\}$

Proof:

$(\Leftarrow)$: $c^T x_0 = c_1 (x_0)_1 + c_2 (x_0)_2 + \dots + c_n (x_0)_n$

(from (ii)) $= (x_0)_1 A_1^T y_0 + (x_0)_2 A_2^T y_0 + \dots + (x_0)_n A_n^T y_0$

(Rearranging) $= (y_0)_1 A_1 x_0 + (y_0)_2 A_2 x_0 + \dots + (y_0)_m A_m x_0$

(from (i)) $= (y_0)_1 b_1 + (y_0)_2 b_2 + \dots + (y_0)_m b_m$

$= b^T y_0$

$(\Rightarrow)$: similar to previous proof.

Primal: maximize $c^T x$

subj. to:

$$\begin{aligned} A_1 x &\sim_1 b_1 \\ A_2 x &\sim_2 b_2 \\ &\vdots \\ A_m x &\sim_m b_m \end{aligned} \qquad \sim_i \in \{\leq, \geq, =\}$$

$$\begin{aligned} x_1 &\sim'_1 0 \\ x_2 &\sim'_2 0 \\ &\vdots \\ x_n &\sim'_n 0 \end{aligned} \qquad \sim'_i \in \{\leq, \geq, \gtrless\}$$

$\gtrless$: unrestricted.

- General form primal will have a corresponding dual.

Theorem: (Complementary slackness for general primal-dual pairs)

Let x_0, y_0 be feasible solutions of primal and dual respectively.

Then: $c^T x_0 = b^T y_0$ iff:

i) $(y_0)_i \, (A_i x_0 - b_i) = 0 \quad \forall i \in \{1, 2, \ldots, m\}$

ii) $(x_0)_j \, (c_j - A_j^T y_0) = 0 \quad \forall i \in \{1, 2, \ldots, n\}$

Proof: Exercise.